Suggested solutions for Homework 1

Disclaimer: There might be some typos or mistakes, let me know if you find any. Therefore, double check!

Exercise 3.5

We need to show that $xy = 0$ if and only if (iff) $x = 0$ or $y = 0$.

First, assume that $x = 0$, then we need to show that $0y = 0$. Using that 0 is the identity of the addition, we get $0y = (0 + 0)y = 0y + 0y$, where we also used the distributive law. Adding $-(0y)$ to both sides of $0y = 0y + 0y$, we get $0 = 0y + 0$. Thus $0 = 0y$. The proof is similar if $y = 0$ is assumed.

For the other direction, assume that $xy = 0$. Also assume that $x \neq 0$. We only need to show that $y = 0$. Multiplying by x^{-1} from the left on both sides of $xy = 0$ gives $y = x^{-1}0 = 0$.

Exercise 3.7

We need to show that $-(xy) = x(-y) = (-x)y$.

First, we pove that $-(xy) = x(-y)$. This is to say that $xy + x(-y) = 0$. However, this is in fact the case, since $xy + x(-y) = x(y + (-y)) = x0 = 0$.

Now, we show in a analogous manner that $-(xy) = (-x)y$. This is the same statement as $xy + (-x)y = 0$. The last statement holds true, since $xy + (-x)y = (x + (-x))y = 0y = 0$.

Exercise 4.7

It is to show that $x^2 + y^2 \geq 2xy$.

First, we observe that $z^2 \geq 0$ for any $z \in \mathbb{R}$. The order axiom tells that exactly one of the following statements hold:

$$z \in P, \quad z = 0, \quad z \in -P.$$

If $z \in P$, then $z^2 = zz \in P$, so $z^2 > 0$. Thus $z^2 \geq 0$. If $z = 0$, then $z^2 = 0$, hence $z^2 \geq 0$. In the last case, we have $-z \in P$, thus $z^2 = -(-(zz)) = (-z)(-z) \in P$. Therefore, also in this case we have $z^2 \geq 0$. Therefore, for any $z \in \mathbb{R}$, $z^2 \geq 0$.

Since $(x - y)^2 = x^2 + y^2 - 2xy$, we have $x^2 + y^2 - 2xy \geq 0$. This is equivalent to $x^2 + y^2 \geq 2xy$.

Exercise 4.9

We show that if $x \leq y + \epsilon$ for every $\epsilon > 0$, then $x \leq y$.

Setting $z = x - y$, it suffices to show that $z \leq 0$, whenever $z \leq \epsilon$ for every $\epsilon > 0$. Assuming the contrary: $z > 0$, we only need to find $\epsilon > 0$ such that $\epsilon < z$ in order to arrive at a contradiction. A plausible candidate for ϵ is $\frac{1}{2}z$.

To be able to define this ϵ, we define $2 := 1 + 1$, which is a positive number, hence non-zero. Since 2 is not zero, it has a multiplicative inverse $\frac{1}{2}$. Now, the definition $\epsilon := \frac{1}{2}z$ can be made.

Since, 2 is positive, also $\frac{1}{2}$ is positive. Therefore, ϵ is positive because both z and $\frac{1}{2}$ are positive. It remains to show that $\epsilon \leq z$. However, since $0 < z$, we have $z < z + z = 2z$. Multiplying by the positive number $\frac{1}{2}$, we arrive at $\epsilon = \frac{1}{2}z < z$. This contradicts the assumption $z > 0$, hence $z \leq 0$.

Exercise 5.1

Let $X \subset \mathbb{R}$ be non-empty and bounded above. Define $a = \mathrm{lub}X$. Let $\epsilon > 0$, we need to show the existence of a

$x \in X$ such that $a - \epsilon < x \leq a$.

The second inequality holds for any $x \in X$ because a is an upper bound for X. Assuming that there is no $x \in X$ such that $a - \epsilon < x$, we conclude that $x \leq a - \epsilon$ for all $x \in X$. This is to say that $a - \epsilon$ is an upper bound for X. This is a contradiction because $a - \epsilon < a$, thus a is not the least upper bound. Consequently, the assumption that there is no $x \in X$ such that $a - \epsilon < x$, is not true.

Exercise 5.7 Let X, Y be non-empty subsets of $\mathbb{R}$ that are bounded above. Define $a = \mathrm{lub}X$, $b = \mathrm{lub}Y$. We show that $a + b = \mathrm{lub}(X + Y)$.

In fact, for any $z \in X + Y$, we can write $z = x + y$ for some $x \in X$ and $y \in Y$. Since a is an upper bound for X and b is an upper bound for Y, we have $z = x + y \leq a + b$. Since $z \in X + Y$ was arbitrary, we conclude that $a + b$ is an upper bound for $X + Y$.

It remains to show that $a + b$ is the least upper bound. Assume that there is a smaller upper bound $c < a + b$. Define $\epsilon = (a + b - c)/2 > 0$. By Exercise 5.1, there is an $x \in X$ such that $a - \epsilon < x$ and a $y \in Y$ such that $b - \epsilon < y$. We have $x + y > a - \epsilon + b - \epsilon = a + b - 2\epsilon = c$. Therefore, c cannot be the an upper bound for $X + Y$.

Exercise 5.8

Let $f : (a, b) \to \mathbb{R}$ be strictly increasing in each $c \in (a, b)$. We show that f is strictly increasing.

Let's assume the contrary that there are $c, d \in (a, b)$ such that $c < d$ and $f(c) \geq f(d)$. Then the set $X = \{x | a < x < d, f(x) \geq f(d)\}$ is non-empty and bounded above. We can define $A = \mathrm{lub}X$.

We know that $A \leq d$. By Exercise 5.1, we know that for all $\epsilon > 0$ there is an $x \in (a, d)$ such that $A - \epsilon < x$ and $f(x) \geq f(d)$. The condition that f is strictly increasing in A provides $\delta > 0$ such that f is strictly increasing on $(A - \delta, A + \delta)$.

We distinguish two cases: $A < d$ and $A = d$. If $A < d$, assume without loss of generality that $\delta < d - A$. We set $\epsilon = \delta/2$ and pick $x \in (A - \epsilon, A]$ such that $f(x) \geq f(d)$. But this contradicts the monotonicity of f in A because $f(A + \epsilon) < f(d) \leq f(x)$. The first equality follows from the definition of A and that $A < A + \epsilon < d$. (Note that here both x and $A + \epsilon$ are in the interval $(A - \delta, A + \delta)$, where f is strictly increasing, also $x < A + \epsilon$).

If $A = d$, then setting $\epsilon = \delta$ and picking $x \in (A - \delta, d)$ such that $f(x) \geq f(d) = f(A)$ yields a contradiction to the strict monotonicity of f in A.