Suggested solutions for Homework 14, part 2

Disclaimer: There might be some typos or mistakes, let me know if you find any. Therefore, double check! The solutions presented here are more concise than what is expected in quizzes and tests.

Notation: $\mathbb{N} = \{\text{positive integers}\}, \mathbb{N}_0 = \{\text{non-negative integers}\}, \mathbb{Z} = \{\text{integers}\}, \mathbb{Q} = \{\text{rational numbers}\}, \mathbb{R} = \{\text{real numbers}\}$ $\mathbb{Q}_{>0}$ stands for positive rationals, similarly for $\mathbb{Q}_{<0}, \mathbb{Q}_{\geq 0}, \mathbb{Q}_{\leq 0}$. For $A \in \mathbb{R}$, $\lfloor A \rfloor$ and $\lceil A \rceil$ are the floor and ceiling of A.

Exercise 59.1 We have to determine for which $p \in \mathbb{R}$ does the Riemann integral $\int_1^\infty \frac{1}{x^p}\mathrm{d}x$ converge.

For $p \leq 0$, the function $f(x) = \int_1^\infty \frac{1}{x^p}\mathrm{d}x \geq 1$ for $x \geq 1$, therefore the integral cannot converge. We will use the integral comparison test to analyze the case $p > 0$. If $p > 0$, the function f is positive and monotonically decreasing, so the integral converges if and only if the series $\sum_{k=1}^\infty f(k)$ converges. We know that $\sum_{k=1}^\infty \frac{1}{k^p}$ converges if and only if $p > 1$, by *Corollary 24.3* and the *Example* before *Theorem 24.2*.

Exercise 59.2 We have to find all pairs $(p, q) \in \mathbb{R}^2$ such that $\int_0^\infty \frac{x^{p-1}}{1+qx}\mathrm{d}x$. We make the observation that the function to be integrated is positive and has no singularities on $x \in [0, \infty]$ if $p \geq 1$, so in these cases we are only concerned about the decay of the function as $x \to \infty$.

Integrability at infinity

The decay should be "at least as fast as of $\frac{1}{x^\alpha}$" for $\alpha > 1$. Ignoring the $+1$ in the denominator for a quick first analysis, we see that the function $\frac{x^{p-2}}{q}$ is integrable on the given interval if $p < 1$. If $q = 0$, then the function x^{p-1} is integrable at infinity, if $p < 0$. So, the decay at infinity is fast enough for integrability if ($q \neq 0$ and $p < 1$) or ($q = 0$ and $p < 0$).

For a more rigorous analysis in the case $q \neq 0$, we can look at the limit of $x^\alpha \frac{x^{p-1}}{1+qx}$ as $x \to \infty$. This goes to ∞ if $\alpha + p - 1 > 1$, goes to 1 if $\alpha + p - 1 = 1$ and goes to zero if $\alpha + p - 1 < 1$. For $p < 1$,we get

$$\int_1^\infty \frac{x^{p-1}}{1+qx} = \int_1^\infty \frac{1}{x^\alpha}\ x^\alpha \frac{x^{p-1}}{1+qx}\mathrm{d}x = \int_1^A \frac{1}{x^\alpha}\ x^\alpha \frac{x^{p-1}}{1+qx}\mathrm{d}x + \int_A^\infty \frac{1}{x^\alpha}\ x^\alpha \frac{x^{p-1}}{1+qx}\mathrm{d}x.$$

The first summand is clearly integrable for $1 < A < \infty$ since the integrand is continuous on the interval $[1, A]$, therefore bounded on any that compact interval. In the second summand $x^\alpha \frac{x^{p-1}}{1+qx}$ goes to zero for $1 < \alpha < 2 - p$. This choice of α is possible since $p < 1$. So choosing A big enough, we have $x^\alpha \frac{x^{p-1}}{1+qx} < 1$ for any $x \geq A$, therefore

$$\int_A^\infty \frac{1}{x^\alpha}\ x^\alpha \frac{x^{p-1}}{1+qx}\mathrm{d}x \leq \int_A^\infty \frac{1}{x^\alpha}\mathrm{d}x < \infty$$

where the integral is finite since $\alpha > 1$. We can similarly show divergence in a rigorous manner for the cases $q \neq 0$ and $p \geq 1$.

Integrability at zero

So far we basically found the domain of (p, q) such that $\int_1^\infty \frac{x^{p-1}}{1+qx}\mathrm{d}x$ converges. However, the function might not be continuous on $[0, 1]$, if it has a singularity in $x = 0$, this happens if $p < 1$ and in that case it is not necessarily true that $\int_0^1 \frac{x^{p-1}}{1+qx}\mathrm{d}x$ is finite. By a change of variable, we see that for $0 < \epsilon < 1$

$$\int_\epsilon^1 x^\alpha \mathrm{d}x = \int_{\frac{1}{\epsilon}}^1 -\frac{1}{y^\alpha}\frac{1}{y^2}\mathrm{d}y = \int_0^{\frac{1}{\epsilon}} \frac{1}{y^{\alpha+2}}\mathrm{d}y.$$

Therefore $\int_0^1 x^\alpha \mathrm{d}x$ is integrable if and only if $\alpha > -1$. (Useful to remember this fact.) In order that the integrand $\int_0^1 \frac{x^{p-1}}{1+qx}\mathrm{d}x$ for $p < 1$ becomes "integrable at 0", we need that it "increases less then x^{-1}" as $x \to 0$. Now, we cannot ignore $+1$ in the denominator, in fact it becomes the dominant factor when $x \approx 0$, and we ignore the summand qx. We conclude that $p - 1 > -1$ is the necessary condition for integrability. Therefore $p > 0$. A rigorous analysis goes as before by looking at the fraction $\frac{1}{x^\alpha}\frac{x^{p-1}}{1+qx}$.

Summary

If $q = 0$, we need $p < 0$ for integrability at infinity, and $p > 0$ for integrability around zero, therefore no choice of $p \in \mathbb{R}$ would imply integrability on $[0, \infty)$.

If $q \neq 0$, we need $p < 1$ for integrability at infinity, and $p > 0$ for integrability at zero. Therefore, the only choices are $p \in (0, 1)$ and $q \in \mathbb{R}$ for which the integrand becomes integrable on $[0, \infty)$.

Exercise 59.6

The idea is here very simple, but somewhat complicated to put down mathematically. Think of a function that takes only the values $\{0, 1\}$ for $x \geq 0$. Such a function if called the indicator function of the set where the function takes the value 1. We take the set to be $A = \cup_{k=1}^{\infty}[k, k + \frac{1}{k^2}]$ The function f to be the indicator function of the set A, $f = \mathbb{1}_A$, i.e. $f(x) = 1$ if $x \in A$, and $f(x) = 0$ if $x \notin A$.

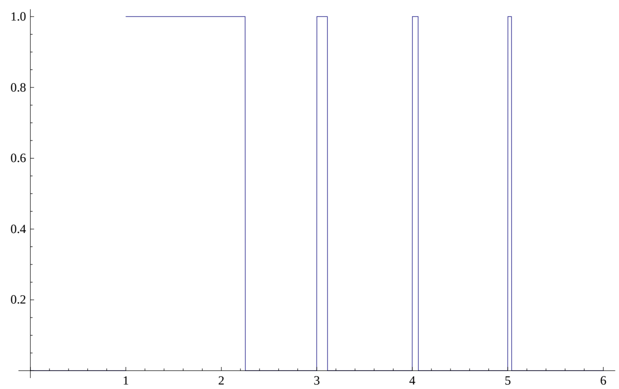


Figure 1: Graph of f on the interval [0,6)

Then $\lim_{x\to\infty} f(x)$ doesn't exist, since for any $M > 0$ there exists $x, y > M$ such that $f(x) = 0$ and $f(y) = 1$. The integral exists however, since

$$\sum_{k=1}^{\lfloor A \rfloor} \frac{1}{k^2} \leq \int_0^{A+1} f(x)\mathrm{d}x \leq \sum_{k=1}^{\lceil A \rceil} \frac{1}{k^2}$$

where $\lfloor A \rfloor$ and $\lceil A \rceil$ are the floor and ceiling of A. Both the left and right hand sides converge to the same number as $A \to \infty$, therefore $\lim_{A\to\infty} \int_0^{A+1} f(x)\mathrm{d}x$ converges to that number.

Exercise 59.10 Let $f \geq 0$ and decreasing on $[1, \infty)$. Then $f(\lceil x \rceil) \leq f(x) \leq f(\lfloor x \rfloor)$ for $x \geq 1$. Thus

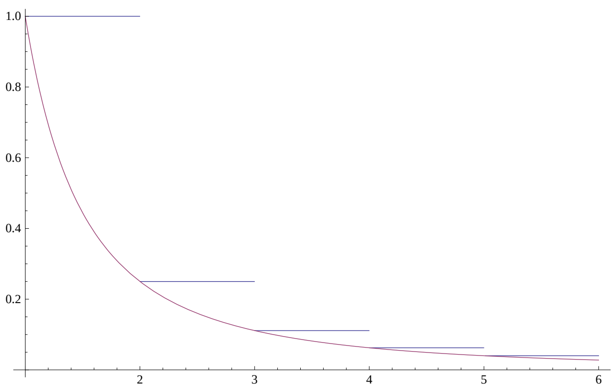
$$\int_1^{m+1} f(x)\mathrm{d}x \leq \sum_{n=1}^{m} f(n) \tag{1}$$

(see Figure 2, a) and

$$\sum_{k=2}^{m} f(n) \leq \int_1^{m} f(x)\mathrm{d}x \tag{2}$$

(see Figure 2, b), therefore

$$
\begin{aligned}
L - \sum_{n=1}^{m} f(n) &\leq L - \int_1^{m+1} f(x)\mathrm{d}x \\
&= L - \left(\int_1^{\infty} f(x)\mathrm{d}x + \int_m^{m+1} f(x)\mathrm{d}\right) + \int_m^{\infty} f(x)\mathrm{d}x \\
&\leq L - \left(\int_1^{\infty} f(x)\mathrm{d}x + f(1)\right) + \int_m^{\infty} f(x)\mathrm{d}x \\
&\leq \left(\sum_{n=2}^{\infty} f(n) - \int_1^{\infty} f(x)\mathrm{d}x\right) + \int_m^{\infty} f(x)\mathrm{d}x \\
&\leq \int_m^{\infty} f(x)\mathrm{d}x.
\end{aligned}
$$



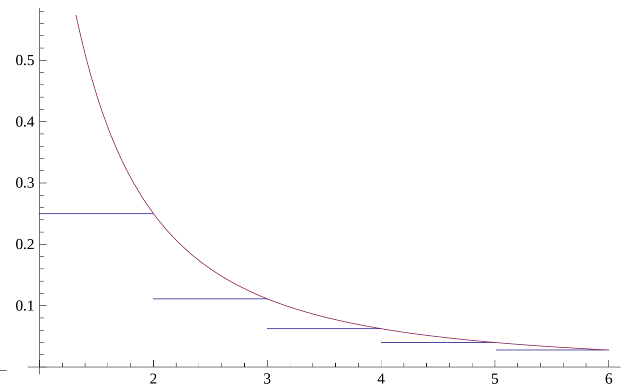


Figure 2: Graph of f on the interval [0,6)

Where the last inequality follows from (2).Since the left hand side if positive, we can take the absolute value on the left hand side.

Similarly, for the other part

$$
\begin{aligned}
L - \int_1^{h} f(x)\mathrm{d}x &\leq L - \sum_{n=2}^{\lfloor h \rfloor} f(n) \\
&\leq L - \left(\sum_{n=2}^{\infty} f(n) + f(\lfloor h \rfloor)\right) + \sum_{n=\lfloor h \rfloor}^{\infty} f(n) \\
&\leq \left(L - \sum_{n=1}^{\infty} f(n)\right) + \sum_{n=\lfloor h \rfloor}^{\infty} f(n) \\
&\leq \sum_{n=\lfloor h \rfloor}^{\infty} f(n).
\end{aligned}
$$

Where the last inequality follows from (1). Since the left hand side if positive, we can take the absolute value on the left hand side.