Suggested solutions for Homework 2

Disclaimer: There might be some typos or mistakes, let me know if you find any. Therefore, double check! The solutions presented here are more concise than what is expected in quizzes and tests.

Notation: $\mathbb{N} = \{\text{positive integers}\}, \mathbb{N}_0 = \{\text{non negative integers}\}, \mathbb{Z} = \{\text{integers}\}, \mathbb{Q} = \{\text{rational numbers}\}, \mathbb{R} = \{\text{real numbers}\}$

Exercise 6.3

Proving the equality by induction over $n \in \mathbb{N}$. The statement is clearly true for $n \in \{0, 1\}$. Assume that the statement is true for n. Then

$$(a+b)^{n+1} = (a+b)(a+b)^n = (a+b)\sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k} = \sum_{k=0}^{n}\binom{n}{k}a^{k+1}b^{n-k} + \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k+1}$$
$$= a^{n+1} + b^{n+1} + \sum_{k=1}^{n}\left(\binom{n}{k-1} + \binom{n}{k}\right)a^k b^{n+1-k}.$$

We only have to show that $\binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k}$ for $1 \leq k \leq n$. This statement however, can be verified by a simple calculation using the definition of the binomial coefficients.

Exercise 6.4

Let $a = \operatorname{lub} X$. Assume for the sake of contradiction that $a \notin X$. However, by Exercise 5.1 there is $x \in X$ such that $x \in (a-1, a)$. Again by Exercise 5.1 there is $y \in X$ such that $y \in (x, a)$. Then we have $x < y$ and $y - x < 1$. This is a contradiction to $y - x \in \mathbb{N}$ that follows from Lemma 6.8.

Exercise 6.5

The problem is that when applying the induction hypothesis to the set $X = \{k-1, m-1\}$, we are not allowed to do so, since $\{k-1, m-1\} \subset \mathbb{N}$ is not guaranteed. The induction hypothesis works only for subsets of $\mathbb{N}$.

Exercise 6.6

Define $X = \{n \in \mathbb{N} | S(n) \text{ is true}\}$. We want to show that $X = \mathbb{N}$. We know that $1 \in X$. Let us define sets $A_n = \{k \in \mathbb{N} | k \leq n\}$. We just observed that $A_1 \subset X$. By (b) we know that whenever $A_n \subset X$, we have $A_{n+1} \subset X$. The first version of mathematical induction implies that $A_n \subset X$ for all $n \in \mathbb{N}$. For any $n \in \mathbb{N}$, we have $n \in A_n$ by definition. Therefore $n \in X$ for any $n \in \mathbb{N}$. This is the same as $\mathbb{N} \subset X$. The other inclusion follows from the definition of X.

Exercise 7.2

We need to show that $\mathbb{Q} \subset \mathbb{R}$ is a field (with the same operations $+$ and $\cdot$ and identities 0, 1 as in the field $\mathbb{R}$). Since $0/1 = 0$, and $1/1 = 1$, we have $0, 1 \in \mathbb{Q}$. We need to show that the addition and the multiplication are maps $\mathbb{Q} \times \mathbb{Q} \to \mathbb{Q}$. Taking $p, q, r, s \in \mathbb{Z}$, we have $p/q + r/s = pq^{-1} + rs^{-1} = pss^{-1}q^{-1} + rqq^{-1}s^{-1} = ps(sq)^{-1} + rq(qs)^{-1} = (ps + rq)(sq)^{-1} = (ps + rq)/(sq)$. Since products of integers are integers, and also $sq \neq 0$, we see that $\mathbb{Q}$ is closed under addition. Similarly, $\mathbb{Q}$ is closed under multiplication.

Actually all the axioms carry over since we know that $\mathbb{R}$ is a field. The only thing that we need to check is that inverses (additive and multiplicative) of rationals are rationals. Let p/q a rational with $p, q \in \mathbb{Z}$, $q \neq 0$, then it is easy to check that $(-p)/q \in \mathbb{Q}$ is its additive inverse and if $p \neq 0$, then q/p is its multiplicative inverse.

Exercise 7.4

Without loss of generality assume $(p/q)^3 = 2$ for $p, q \in \mathbb{N}$, such that at most one of the numbers p, q are even. Then $p^3 = 2q^3$. Therefore p is even, thus $p = 2n$ for some $n \in \mathbb{N}$. We have $8n^3 = 2q^3$, so $4n^3 = q^3$. This means that q is even (this follows from the Euclidean algorithm), which is a contradiction.

Exercise 7.5

Let's now prove (a) for a fixed $m \in \mathbb{N}$ using induction over $n \in \mathbb{N}$. If $n = 1$ then we have $x^m x = x^{m+1}$. Assume that it holds for $n \in \mathbb{N}$, then $x^m x^{n+1} = x^m x^n x = x^{m+n} x = x^{m+n+1}$.

Now (a) obviously holds if either m or n is zero. For both n, m negative, we can use $x^{-(m+n)} = x^{-m}x^{-n}$ and multiply with $x^m x^n x^{m+n}$ on both sides. Now assume that $m < n$ are positive. We know that $x^n = x^{n-m}x^m$, thus $x^{-m}x^n = x^{n-m}$. If $n < m$ just take the multiplicative inverses on both sides.

Part (b) follows from (a), since $x^n x^{-n} = x^0 = 1$

For part (c) and (d), we do induction over $n \in \mathbb{N}$ then consider the multiplicative inverses. Part (e) follows from parts (c) and (b). Part (f) is done by induction over n.

For part (g), we notice that by part (f), we have $1^{m-n} < x^{m-n}$. Since $x > 0$, also $x^n > 0$. Now multiplying by x^n on both sides of $1 < x^{m-n}$, we obtain $x^n < x^m$.

Exercise 7.7

Let $X = \{x \in \mathbb{R} | x \geq 0 \text{ and } x^n \leq a\}$ and $b = \text{lub}X$. Assume that $a < b^n$. Define $\delta = b^n - a > 0$. Just like in the proof of Theorem 7.5, pick m to be so big such that

$$\binom{n}{k} b^k \frac{1}{m^{n-k}} < \frac{\delta}{n} \quad \text{for all } k \in \{0, 1..., n-1\}.$$

Then we will also have

$$(-1)^{n-k} \binom{n}{k} b^k \frac{1}{m^{n-k}} > -\frac{\delta}{n} \quad \text{for all } k \in \{0, 1..., n-1\}.$$

This is true because in the cases where $(-1)^{n-k} = 1$, the left hand side is positive and the right hand side is negative, and in the cases where $(-1)^{n-k} = -1$, this is a consequence of the previous inequalities. Computing $(b - \frac{1}{m})^n$ using the binomial formula, gives

$$\begin{aligned}
\left(b - \frac{1}{m}\right)^n &= \sum_{k=0}^{n-1} \binom{n}{k} b^k \frac{1}{(-m)^{n-k}} + b^n \\
&= \sum_{k=0}^{n-1} (-1)^{n-k} \binom{n}{k} b^k \frac{1}{m^{n-k}} + b^n \\
&> \sum_{k=0}^{n-1} -\frac{\delta}{n} + b^n \\
&= -\delta + b^n = a.
\end{aligned}$$

We show that $\tilde{b} = b - \frac{1}{m}$ is an upper bound of X. Let $x \in X$ and assume that $x > \tilde{b}$, then $x^n > \tilde{b}^n$. From $x \in X$ and the computation above, we know that $x^n \leq a < \tilde{b}^n$, which leads us to a contradiction. Therefore, b is not the least upper bound of X, thus $b^n \leq a$.

Exercise 7.8

Let $a > 0$ and $n \in \mathbb{N}$. We need to show that there is a unique $b > 0$ such that $b^n = a$. (This unique b is called the n-th root of a). We already showed that there exists such b. Assume that there exist $b, \tilde{b} > 0$, $b \neq \tilde{b}$ such that $b^n = a$ and $\tilde{b}^n = a$. Without loss of generality assume that $b < \tilde{b}$. Then $a = b^n < \tilde{b}^n = a$ that is a contradiction, hence $b = \tilde{b}$.

Exercise 7.10

Assume that $a = (a_1, a_2),\ b = (b_1, b_2),\ c = (c_1, c_2) \in \mathbb{Q}^2$ are the vertices of an equilateral triangle. I.e.

$$\begin{aligned}(a_1 - b_1)^2 + (a_2 - b_2)^2 &= d\\ (b_1 - c_1)^2 + (b_2 - c_2)^2 &= d\\ (c_1 - a_1)^2 + (c_2 - a_2)^2 &= d\end{aligned}$$

for some fixed $d > 0$ (note that d is the square of the side length). By translating $a \mapsto a - a$, $b \mapsto b - a$, $c \mapsto c - a$, we can assume that $a = 0$. So, setting $a = 0$, we have

$$\begin{aligned}b_1^2 + b_2^2 &= d\\ (b_1 - c_1)^2 + (b_2 - c_2)^2 &= d\\ c_1^2 + c_2^2 &= d\end{aligned}$$

By multiplying out the brackets in the second equation and substituting d from the two other ones, we get $d = b_1^2 + b_2^2 + c_1^2 + c_2^2 - 2(b_1c_1 + b_2c_2) = 2d - 2(b_1c_1 + b_2c_2)$. Thus

$$2(b_1c_1 + b_2c_2) = d.$$

Using the identity $(a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 + (ad - bc)^2$, we have $d^2 = (b_1^2 + b_2^2)(c_1^2 + c_2^2) = (b_1c_1 + b_2c_2)^2 + (b_1c_2 - c_1b_2)^2$, and therefore $d^2 = \frac{d^2}{4} + (b_1c_2 - c_1b_2)^2$. This is the same as $3d^2 = 4(b_1c_2 - c_1b_2)^2$. Since $d \in \mathbb{Q}$, $d > 0$, and the right hand side is the square of a rational number, we conclude that 3 is the square of a rational number. This is a contradiction since $\sqrt{3} \notin \mathbb{Q}$ by the same method as in Exercise 7.4.