Suggested solutions for Homework 3

Disclaimer: There might be some typos or mistakes, let me know if you find any. Therefore, double check! The solutions presented here are more concise than what is expected in quizzes and tests.

Notation: $\mathbb{N} = \{\text{positive integers}\}, \mathbb{N}_0 = \{\text{non negative integers}\}, \mathbb{Z} = \{\text{integers}\}, \mathbb{Q} = \{\text{rational numbers}\}, \mathbb{R} = \{\text{real numbers}\}$

$X \approx Y$ means X and Y are equivalent (they have the same cardinality). For a finite set X, let $|X| \in \mathbb{N}_0$ be the number of elements of X.

Exercise 8.2 We show $X \approx Y$ then we have $\mathcal{P}(X) \approx \mathcal{P}(X)$.

Since $X \approx Y$, there is a bijection $f : X \to Y$. We define the map $\bar{f} : \mathcal{P}(X) \to \mathcal{P}(Y)$ by $\bar{f}(Z) = \{f(z)| \ z \in Z\}$. Often the map $\bar{f}$ is denoted by f which might create some ambiguity in our case, so we bare with $\bar{f}$.

- **The map $\bar{f}$ is well-defined** since for $z \in Z \subset X$, $f(z)$ is defined and $f(z) \in Y$ such that $\bar{f}(Z) \subset Y$.
- **The map $\bar{f}$ is one-to-one**: Pick $Z, W \in \mathcal{P}(X)$ such that $Z \neq W$. Then there is an element $z \in Z \triangle W$ (the symmetric difference). Assume without loss of generality that $z \in Z$ and $z \notin W$. In this case $f(z) \in \bar{f}(Z)$. Also $f(z) \notin \bar{f}(W)$, because otherwise there was a $w \in W$ such that $f(w) = f(z)$, which means that $w = z$ since f is one-to-one. This contradicts $z \in Z$, $z \notin W$, so $f(z) \notin \bar{f}(W)$. Consequently, $\bar{f}(Z) \neq \bar{f}(W)$, this proves that $\bar{f}$ is one-to-one.
- **The map $\bar{f}$ is onto**: Let's pick $Z \in \mathcal{P}(Y)$. Then for the set $W = \{f^{-1}(z)|z \in Z\} \subset X$, we have $\bar{f}(W) = Z$. This means that $\bar{f}$ is onto $\mathcal{P}(Y)$.

Exercise 8.4

We show that X is infinite if and only if $X \approx Y$ for some proper subset $Y \subset X$.

First, we show that if X is finite, then there is no proper subset of X that is equivalent to X. If X is finite then $|X| = n \in \mathbb{N}_0$. If $n = 0$ then X has no proper subsets, so it cannot be equivalent to any. If $n \in \mathbb{N}$, then any proper subset $Y \subset X$ has $|Y| < n$. Since X, Y are both finite with different number of elements, they cannot be equivalent.

Assume that X is countably infinite. Then there is a bijection $f : \mathbb{N} \to X$. Let's define the map $g : X \to X$ by $g(x) = f(f^{-1}(x) + 1)$. This means that $f(n) \mapsto f(n + 1)$.

- **The map g is well-defined** since for $x \in X$, $f^{-1}(x) \in \mathbb{N}$, thus $f^{-1}(x) + 1 \in \mathbb{N}$.
- **The map g is one-to-one** since for $x, y \in X$ such that $x \neq y$, we have $f^{-1}(x) \neq f^{-1}(y)$ because f, thus f^{-1} is one-to-one. Therefore also $f^{-1}(x) + 1 \neq f^{-1}(y) + 1$, which in turn gives $f(f^{-1}(x) + 1) \neq f(f^{-1}(y) + 1)$ using that f is one-to-one. The latter means that $g(x) \neq g(y)$, i.e. g is one-to-one.
- **The image $g(X) \subset X$ is a proper subset** since $f(1) \notin g(X)$.

Now, assume that X is uncountable. Then there is a one-to-one map $f : \mathbb{N} \to X$. Then $f^{-1} : f(\mathbb{N}) \to \mathbb{N}$ exists. We define $g : X \to X$ by

$$g(x) = \begin{cases} f(f^{-1}(x)+1) & \text{if } x \in f(\mathbb{N}) \\ x & \text{if } x \notin f(\mathbb{N}). \end{cases}$$

Similarly as in the case where X countably infinite, we show that $g : X \to g(X)$ is one-to-one and onto the proper subset $g(X) \subset X$.

Exercise 9.1 Using that subsets of $\mathbb{N}$ are countable, we need to show that a subset $A \subset B$ of a countable set B is countable.

If B is finite, then any subset of B is finite. If B is countably infinite, there is a bijection $f : B \to \mathbb{N}$. The restriction of f to A is a bijection $f|_A : A \to f(A)$. Since $f(A) \subset \mathbb{N}$, we know that $f(A)$ is countable. If $f(A)$ is finite, then A is finite and thus countable. If $f(A)$ is countably infinite, there is a bijection $g : f(A) \to \mathbb{N}$. In this case $g \circ f|_A : A \to \mathbb{N}$ is a bijection, therefore A is countably infinite.

Exercise 9.3 easily follows from Exercise 9.4 because $B \subset A \cup B$.

Exercise 9.4

Assume that $A \subset B$, and A is uncountable. If B was countable, then A was countable by Exercise 9.1. Therefore B is uncountable.

Exercise 9.7

We define $\mathcal{P}_n = \{\sum_{k=0}^n q_k x^k | q_k \in \mathbb{Q}, q_n \neq 0\}$ the set of polynomials of degree n with rational coefficients. There is a natural bijection $f : \mathbb{Q}^n \times \mathbb{Q} \setminus \{0\} \to \mathcal{P}_n$, defined by $f(q_0, q_1, ..., q_n) \mapsto \sum_{k=0}^n q_k x^k$. Since $\mathbb{Q}^n \times \mathbb{Q} \setminus \{0\} \subset \mathbb{Q}^{n+1}$, the set $\mathbb{Q}^n \times \mathbb{Q} \setminus \{0\}$ is countable, therefore $\mathcal{P}_n$ is also countable.

Exercise 9.8

The set of all polynomials with rational coefficients $\mathcal{P} = \cup_{n \in \mathbb{N}_0} \mathcal{P}_n$ is a countable union of countable sets, therefore countable.

Exercise 9.9

(a) Let $q \in \mathbb{Q}$, then $p_1(x) = x - q$ has root q, and for $q > 0$, the polynomial $p_2(x) = x^2 - q$ has root $\sqrt{q}$. Thus $q, \sqrt{q}$ are algebraic.

(b) We need to show that r is algebraic iff it is the root of a polynomial with integer coefficients.

If r is the root of a polynomial with integer coefficients, then it is algebraic by definition, noting that $\mathbb{Z} \subset \mathbb{Q}$. Assume that r algebraic, i.e. there is a $p \in \mathcal{P}_n$ for some $n \in \mathbb{N}_0$ such that $p(r) = 0$. If $p(x) = \sum_{k=0}^n q_k x^k$ where $q_k = \frac{s_k}{t_k}$ for $s_k, t_k \in \mathbb{Z}$, $t_k \neq 0$, then defining $N = \prod_{k=0}^n t_k \neq 0$ and the polynomial Np, clearly $Np(r) = 0$. However, Np is a polynomial with integer coefficients.

(c) We need to show that the set of algebraic numbers is countable. Noting that any $p \in \mathcal{P}_n$ has at most n roots, the set $\cup_{p \in \mathcal{P}_n} \mathrm{roots}(p)$ is a countable set, because it is the countable union of finite sets. Here, $\mathrm{roots}(p)$ denotes the set of the roots of p. Also $A = \cup_{n \in \mathbb{N}_0} \cup_{p \in \mathcal{P}_n} \mathrm{roots}(p)$ is a countable union of countable sets, therefore countable. However, A is exactly the set of all algebraic numbers.

Let T be the set of transcendent numbers, then $A \cup T = \mathbb{R}$, by definition of the transcendent numbers. If T was countable, then $\mathbb{R}$ was countable, therefore T is uncountable. An uncountable set cannot be empty, therefore $T \neq \emptyset$, so transcendent numbers exist.