Suggested solutions for Homework 4

Disclaimer: There might be some typos or mistakes, let me know if you find any. Therefore, double check! The solutions presented here are more concise than what is expected in quizzes and tests.

Notation: $\mathbb{N} = \{\text{positive integers}\}, \mathbb{N}_0 = \{\text{non-negative integers}\}, \mathbb{Z} = \{\text{integers}\}, \mathbb{Q} = \{\text{rational numbers}\}, \mathbb{R} = \{\text{real numbers}\}$

$X \approx Y$ means X and Y are equivalent (they have the same cardinality). For a finite set X, let $|X| \in \mathbb{N}_0$ be the number of elements of X.

Exercise 10.3 It is to show that $\lim_{n\to\infty} \frac{1}{n+2} = 0$.

Fix $\epsilon > 0$. Denote $a_n = \frac{1}{n+2}$. We need to find $N \in \mathbb{N}$ such that

$$|a_n - 0| = \left|\frac{1}{n+2}\right| < \epsilon, \quad \text{whenever } n \geq N.$$

By the Archimedean property of $\mathbb{R}$, pick $N \in \mathbb{N}$ so big such that $\frac{1}{\epsilon} < N + 2$. Then for all $n \geq N$, we have $\frac{1}{\epsilon} < N + 2 \leq n + 2$. Therefore $\left|\frac{1}{n+2}\right| < \epsilon$.

Exercise 10.6 It is to show that $\lim_{n\to\infty} \frac{2n}{n+2} = 2$.

Fix $\epsilon > 0$. Denote $a_n = \frac{2n}{n+2}$. We need to find $N \in \mathbb{N}$ such that

$$|a_n - 2| = \left|\frac{2n}{n+2} - 2\right| < \epsilon, \quad \text{whenever } n \geq N.$$

Compute

$$a_n = \frac{2n}{n+2} = \frac{2(n+2) - 4}{n+2} = 2 - \frac{4}{n+2}$$

By the Archimedean property of $\mathbb{R}$, pick $N \in \mathbb{N}$ so big such that $\frac{4}{\epsilon} < N + 2$. Then for all $n \geq N$, we have $\frac{4}{\epsilon} < N + 2 \leq n + 2$. Therefore

$$|a_n - 2| = \left|\frac{4}{n+2}\right| < \epsilon, \quad \text{whenever } n \geq N.$$

In the rest of the homework, we will use the following lemma several times.

Lemma (Reverse triangle inequality)**.** *For real numbers x, y, the inequality $||x| - |y|| \leq |x - y|$ holds.*

Proof. By the usual triangle inequality $|x| = |(x - y) + y| \leq |x - y| + |y|$. Hence $|x| - |y| \leq |x - y|$. By switching x and y, we have $|y| - |x| \leq |y - x|$, therefore also $|y| - |x| \leq |x - y|$. In conclusion $||x| - |y|| \leq |x - y|$. $\square$

Exercise 10.10 It is to show that $a_n = n + \frac{1}{n}$ has no limit.

Assume that $\lim_{n\to\infty} a_n = L \in \mathbb{R}$. For all $n \geq |L| + 1$, we have

$$|a_n - L| = \left|n + \frac{1}{n} - L\right| \geq \left|n + \frac{1}{n}\right| - |L| = n + \frac{1}{n} - |L| \geq 1 + \frac{1}{n} > 1,$$

where we used the reverse triangle inequality in the first inequality. Thus $|a_n - L| > 1$ whenever $n \geq |L| + 1$, i.e. a_n cannot converge to L.

Exercise 10.12 Let $\lim_{n\to\infty} a_n = L$. We need to prove that $\lim_{n\to\infty} |a_n| = |L|$.

Fix $\epsilon > 0$. Since $\lim_{n\to\infty} a_n = L$, there is an $N \in \mathbb{N}$ such that

$$|a_n - L| < \epsilon, \quad \text{whenever } n \geq N.$$

By the reverse triangle inequality,

$$||a_n| - |L|| \leq |a_n - L| < \epsilon, \quad \text{whenever } n \geq N.$$

Exercise 11.4 We need to find the rule for $\{a_n\}_{n=1}^{\infty}$ and find the formula for the subsequence $\{a_{f(n)}\}_{n=1}^{\infty}$.

We can write $a_n = \frac{b_n}{c_n}$, where $c_n = 2^{n-1}$. Observing the sequence b_n, we find the recursive rule that $b_1 = 1$ and $b_{n+1} = 2b_n + (-1)^n$. We wish to find a closed formula for b_n, so let us substitute the formula for b_n in the formula for b_{n+1}, we get

$$\begin{aligned} b_{n+1} &= 2b_n + (-1)^n \\ &= 2(2b_{n-1} + (-1)^{n-1}) + (-1)^n \\ &= 4b_{n-1} + 2(-1)^{n-1} + (-1)^n. \end{aligned}$$

Repeating this procedure i.e. substituting the rule for b_{n-1} as next, we see that

$$b_{n+1} = 8b_{n-2} + 4(-1)^{n-2} + 2(-1)^{n-1} + (-1)^n,$$

If we repeat until we get to b_1 on the right hand side, we have

$$\begin{aligned} b_{n+1} &= 2^n b_1 + 2^{n-1}(-1)^1 + 2^{n-2}(-1)^2 + \cdots + 2^{n-2}(-1)^2 + 2^0(-1)^n \\ &= \sum_{k=0}^{n} 2^k (-1)^{n-k} \\ &= (-1)^n \sum_{k=0}^{n} 2^k (-1)^{-k} \\ &= (-1)^n \sum_{k=0}^{n} 2^k (-1)^{k} \\ &= (-1)^n \sum_{k=0}^{n} (-2)^k \\ &= (-1)^n \frac{1-(-2)^{n+1}}{1-(-2)} = (-1)^n \frac{1-(-2)^{n+1}}{3}. \end{aligned}$$

Here we used the formula $\sum_{k=0}^{n} \gamma^k = \frac{1-\gamma^{n+1}}{1-\gamma}$, for $\gamma \neq 0$. Therefore, $b_n = (-1)^{n-1} \frac{1-(-2)^n}{3}$ and $a_n = \frac{b_n}{c_n} = (-1)^{n-1} \frac{1-(-2)^n}{3\ 2^{n-1}} = \frac{1-(-2)^n}{3\ (-2)^{n-1}}$.

Obviously, the subsequence $\{a_{f(n)}\}_{n=1}^{\infty}$ picks every second element of the sequence $\{a_n\}_{n=1}^{\infty}$ beginning with the 2nd, therefore, $f(n) = 2n$ and $a_{f(n)} = \frac{1-(-2)^{2n}}{3\ (-2)^{2n-1}}$.

Exercise 11.8 Assume that $\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} a_{2n-1} = L$. We need to show that $\lim_{n\to\infty} a_n = L$.

Fix $\epsilon > 0$ and pick N_1 so big such that

$$|a_{2n} - L| < \epsilon, \quad \text{whenever } n \geq N_1.$$

Also pick N_2 so big such that

$$|a_{2n-1} - L| < \epsilon, \quad \text{whenever } n \geq N_2.$$

Define $N = 2\max(N_1, N_2)$. Then for any $m \geq N$, we have $m \geq 2N_1$ and $m \geq 2N_2$.

For such $m \geq N$ if m even then $m = 2n$ for some $n \geq N_1$. If $m \geq N$ is odd then $m = 2n - 1$ for some $n > N_2$. Therefore

$$|a_m - L| < \epsilon, \quad \text{whenever } m \geq N.$$

Exercise 11.9 Let $\{a_n\}_{n=1}^{\infty}$ have finite range $\{L_1, \cdots L_K\}$ for $K \in \mathbb{N}$ and $L_k \in \mathbb{R}$ for all $k \in \{1, 2, \cdots, K\}$. We need to show that $\{a_n\}_{n=1}^{\infty}$ has a subsequence which converges.

Since $a_n \in \{L_1, \cdots L_K\}$ for all $n \in \mathbb{N}$, there is a $k \in \{1, 2, \cdots, K\}$ such that $a_n = L_k$ for infinitely many n. If this wasn't true, then each value L_k would appear in the sequence only finitely many times, but that contradicts the sequence being infinitely long. Let us fix that k for which $a_n = L_k$ for infinitely many n.

Define the index set $A = \{n \in \mathbb{N} \mid a_n = L_k\}$. Note that A is unbounded. We define the function $f : \mathbb{N} \to A$ recursively as follows:

$$f(1) := \min A$$
$$f(m+1) := \min\{n \in A \mid n > f(m)\}.$$

Then f is increasing with values in A. Thus $\{a_{f(n)}\}_{n=1}^{\infty}$ is a subsequence such that $a_{f(n)} = L_k$ for all $n \in \mathbb{N}$. This means that $\{a_{f(n)}\}_{n=1}^{\infty}$ is a constant sequence L_k, therefore it converges to L_k.

Exercise 11.11 Let $\{a_n\}_{n=1}^{\infty}$ be a real valued sequence and $f : \mathbb{N} \to \mathbb{N}$ increasing. Then $\{a_{f(n)}\}_{n=1}^{\infty}$ is a subsequence of $\{a_n\}_{n=1}^{\infty}$. Furthermore, if $g : \mathbb{N} \to \mathbb{N}$ is increasing, then $\{a_{f(g(n))}\}_{n=1}^{\infty}$ is a subsequence of $\{a_{f(n)}\}_{n=1}^{\infty}$.

However $f(g(n)) = f \circ g(n)$, where $f \circ g : \mathbb{N} \to \mathbb{N}$ is an increasing function, therefore $\{a_{f \circ g(n)}\}_{n=1}^{\infty}$ is in fact a subsequence of $\{a_n\}_{n=1}^{\infty}$.

Exercise 11.12 We need to show that the set of subsequences of $\{\frac{1}{n}\}_{n=1}^{\infty}$ is uncountable.

The map $f : \mathbb{N} \to \mathbb{R}$, $f(n) = \frac{1}{n}$ is one-to-one. This also means that for g, h: $\mathbb{N} \to \mathbb{N}$ increasing $f \circ g = f \circ h$ iff $g = h$. In fact, if $g = h$, then obviously $f \circ g = f \circ h$. On the other hand, if $f \circ g = f \circ h$ then for all $n \in \mathbb{N}$, we have $f \circ g(n) = f \circ h(n)$, thus $g(n) = h(n)$ because f is one-to-one. But this means that $g = h$.

This means that there is a bijection between the sets A and B

$$A = \{g : \mathbb{N} \to \mathbb{N} \mid g \text{ is increasing }\}$$
$$B = \{f \circ g \mid g : \mathbb{N} \to \mathbb{N} \text{ is increasing }\}.$$

Note that B is the set of all subsets of $\{\frac{1}{n}\}_{n=1}^{\infty}$ and A is the set of increasing sequences with values in $\mathbb{N}$. Since $A \approx B$, it suffices to show that A is uncountable.

Assume for the sake of contradiction that A is countable. Then the elements of A can be listed $A = \{a^{(1)}, a^{(2)}, a^{(3)}, \cdots\}$. Note that each $a^{(i)}$ is a sequence $\{a_n^{(i)}\}_{n=1}^{\infty}$. Let us define an increasing $b : \mathbb{N} \to \mathbb{N}$ recursively as follows

$$b_1 := a_2^{(1)}$$
$$b_{m+1} := \min\{n \in \mathbb{N} \mid n > \max(a_{m+1}^{(m+1)}, b_m)\}.$$

Then for any $m \in \mathbb{N}$, we have $b_m > a_m^{(m)}$, thus $b \notin A$ which is a contradiction. Therefore A is uncountable.

Exercise 12.2 Dividing by n^2 in both the numerator and the denominator in the first step, we obtain

$$\begin{aligned}\lim_{n\to\infty}\frac{2n^2+n+3}{n^2+1} = \lim_{n\to\infty}\frac{2+\frac{1}{n}+\frac{3}{n^2}}{1+\frac{1}{n^2}} &= \frac{\lim_{n\to\infty}\left(2+\frac{1}{n}+\frac{3}{n^2}\right)}{\lim_{n\to\infty}\left(1+\frac{1}{n^2}\right)} \\ &= \frac{\lim_{n\to\infty}2+\lim_{n\to\infty}\frac{1}{n}+\lim_{n\to\infty}\frac{3}{n^2}}{\lim_{n\to\infty}1+\lim_{n\to\infty}\frac{1}{n^2}} \\ &= \frac{2+0+3\left(\lim_{n\to\infty}\frac{1}{n}\right)\left(\lim_{n\to\infty}\frac{1}{n}\right)}{1+\left(\lim_{n\to\infty}\frac{1}{n}\right)\left(\lim_{n\to\infty}\frac{1}{n}\right)} \\ &= \frac{2+0+3\cdot 0\cdot 0}{1+0\cdot 0} = 2,\end{aligned}$$

where we repeatedly used that $\lim_{n\to\infty}\frac{1}{n}=0$.

Exercise 12.4 Using the identity $(a-b)(a+b)=a^2-b^2$ we obtain $(\sqrt{n+1}-\sqrt{n})(\sqrt{n+1}+\sqrt{n})=n+1-n=1$, therefore

$$\lim_{n\to\infty}\sqrt{n+1}-\sqrt{n}=\lim_{n\to\infty}\frac{1}{\sqrt{n+1}+\sqrt{n}}.$$

We claim that this limit goes to 0. In fact, fix $\epsilon>0$ and pick $N\in\mathbb{N}$ so big such that $\frac{1}{\epsilon^2}<N$. Then for any $n\geq N$, we have

$$\left|\frac{1}{\sqrt{n+1}+\sqrt{n}}-0\right| = \frac{1}{\sqrt{n+1}+\sqrt{n}} \leq \frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{N}} < \epsilon.$$

Exercise 12.6 Let $a_n\geq 0$ for $n\in\mathbb{N}$ and $\lim_{n\to\infty}a_n=L$. We show that $\lim_{n\to\infty}\sqrt{a_n}=\sqrt{L}$.

If $a_n\geq 0$, then it is easy to see that $L\geq 0$. Let is consider the cases that $L=0$ and $L>0$.

In the case $L=0$, fix $\epsilon>0$, pick $N\in\mathbb{N}$ so big such that $|a_n|<\epsilon^2$ for all $n\geq N$. Then $|\sqrt{a_n}-0|=\sqrt{a_n}<\epsilon$ for all $n\geq N$.

In the case $L>0$, we have $(\sqrt{a_n}-\sqrt{L})(\sqrt{a_n}+\sqrt{L})=a_n-L$, therefore

$$|\sqrt{a_n}-\sqrt{L}|=\frac{|a_n-L|}{\sqrt{a_n}+\sqrt{L}}\leq\frac{|a_n-L|}{\sqrt{L}}.$$

Fix $\epsilon>0$ and pick $N\in\mathbb{N}$ such that $|a_n-L|<\epsilon\sqrt{L}$ whenever $n\geq N$. Then for any $n\geq N$, we have $|\sqrt{a_n}-\sqrt{L}|<\epsilon$.

Exercise 12.7 Let $a_n\neq 0$ assume that $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=0$. We show that then $\lim_{n\to\infty}a_n=0$.

Since $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=0$, there exists a $N\in\mathbb{N}$ such that for all $n\geq N$, we have $\left|\frac{a_{n+1}}{a_n}\right|<\frac{1}{2}$, i.e. $|a_{n+1}|<\frac{1}{2}|a_n|$.

The inequality $|a_{n+1}|<\frac{1}{2}|a_n|$ holds for any $n\geq N$, therefore by iteration, for any $k\in\mathbb{N}$, we have

$$|a_{N+k}|<\left(\frac{1}{2}\right)^k|a_N|.$$

Fix $\epsilon>0$, pick $K\in\mathbb{N}$ so big such that $\left(\frac{1}{2}\right)^k|a_N|<\epsilon$. Then for all $k\geq K$, we have $|a_{N+k}|<\epsilon$. Therefore for all $n\geq\widetilde{N}:=N+K$, we have $|a_n|<\epsilon$. This is to say that $\lim_{n\to\infty}a_n=0$.