Suggested solutions for Homework 5

Disclaimer: There might be some typos or mistakes, let me know if you find any. Therefore, double check! The solutions presented here are more concise than what is expected in quizzes and tests.

Notation: $\mathbb{N} = \{\text{positive integers}\}, \mathbb{N}_0 = \{\text{non-negative integers}\}, \mathbb{Z} = \{\text{integers}\}, \mathbb{Q} = \{\text{rational numbers}\}, \mathbb{R} = \{\text{real numbers}\}$

$X \approx Y$ means X and Y are equivalent (they have the same cardinality). For a finite set X, let $|X| \in \mathbb{N}_0$ be the number of elements of X.

Exercise 13.3 Defining $a_n := (-1)^n$, $b_n = 1$ clearly $|a_n| \leq 1$ is bounded and $\lim_{n\to\infty} b_n = 1$ is convergent. However, $a_n + b_n = (-1)^n + 1$ and $a_n b_n = (-1)^n$ are divergent. The idea to prove this is the same in both cases, we show that $c_n = a_n b_n = (-1)^n$ is divergent. Note that the sequence takes both values -1 and 1 infinitely often. Assume that $\lim_{n\to\infty} c_n = L$. Then there is a $N \in \mathbb{N}$ such that $|c_n - L| < 1$ for all $n \geq N$. This is a contradiction since there are $k \geq N$ and $l \geq N$ such that $c_k = 1$ and $c_l = -1$. Then $2 = |c_k - c_l| = |(c_k - L) - (c_l - L)| \leq |c_k - L| + |c_l - L| < 1 + 1 = 2$ which is a contradiction.

Exercise 13.4 Define $a_n = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n}$. Then $|a_n| = \left|\frac{1}{2}\right| \cdot \left|\frac{3}{4}\right| \cdot \left|\frac{5}{6}\right| \cdots \left|\frac{2n-1}{2n}\right| < 1$.

Exercise 15.1 Let $\lim_{n\to\infty} a_n = \infty$ and pick a subsequence $\{a_{f(n)}\}_{n=1}^{\infty}$. Fix $M > 0$. By definition of divergence to infinity, there is a $N \in \mathbb{N}$ such that $a_n > M$ whenever $n \geq N$. Since $f : \mathbb{N} \to \mathbb{N}$ is increasing, we can show by induction that for all n, we have $n \leq f(n)$. This means that $a_{f(n)} > M$ whenever $n \geq N$, proving that $\lim_{n\to\infty} a_{f(n)} = \infty$.

Exercise 15.2 Let $\lim_{n\to\infty} a_n = \infty$ and $|b_n| \leq K$. Fix $M > 0$. By definition of divergence to infinity, there exists $N \in \mathbb{N}$ such that $a_n > M + K$ whenever $n \geq N$. Using that $b_n \leq |b_n| \leq K$, we obtain

$$a_n + b_n \geq a_n - |b_n| \geq a_n - K > M,$$

where the last inequality holds whenever $n \geq N$.

Exercise 15.6 We need to show that $\lim_{n\to\infty} a_n = \infty$ iff there exists $N \in \mathbb{N}$ such that $0 < a_n$ for all $n \geq N$ and $\lim_{n\to\infty} \frac{1}{a_n} = 0$.

Assume $\lim_{n\to\infty} a_n = \infty$ and fix $\epsilon > 0$. Then there exists a $N \in \mathbb{N}$ such that $a_n > \frac{1}{\epsilon}$ whenever $n \geq N$. Thus also $\left|\frac{1}{a_n} - 0\right| = \frac{1}{a_n} < \epsilon$ whenever $n \geq N$, proving that $a_n > 0$ whenever $n \geq N$ and $\lim_{n\to\infty} \frac{1}{a_n} = 0$.

Now assume that there is $N \in \mathbb{N}$ such that $a_n > 0$ whenever $n \geq N$ and $\lim_{n\to\infty} \frac{1}{a_n} = 0$. Call $b_n = \frac{1}{a_n}$ and fix $M > 0$. We find $\widetilde{N} \in \mathbb{N}$ such that $|b_n| < \frac{1}{M}$ whenever $n \geq \widetilde{N}$. Note that for $n \geq \max(N, \widetilde{N})$, we have $0 < b_n = |b_n| < \frac{1}{M}$. This shows that $M < a_n$ whenever $n \geq \max(N, \widetilde{N})$, proving that $\lim_{n\to\infty} a_n = \infty$.

Exercise 16.3 Let $\{a_n\}$ be increasing and not bounded. Note that $a_1 \leq a_n$ for all $n \in \mathbb{N}$, i.e. the sequence is bounded below by a_1. Fix $M > 0$, then there is an $N \in \mathbb{N}$ such that $a_N > M$, this N exists since otherwise $\{a_n\}$ would be bounded. By monotonicity, $a_n \geq a_N > M$ for all $n \geq N$.

Exercise 16.5 Note that $\{\left(1 + \frac{1}{n^2}\right)^{n^2}\}$ is a subsequence of $\{\left(1 + \frac{1}{n}\right)^n\}$, so it has the same limit e.

For (b) let us write $\left(1 + \frac{1}{n+1}\right)^n = \left(1 + \frac{1}{n+1}\right)^{n+1} \cdot \left(1 + \frac{1}{n+1}\right)^{-1}$. The first factor converges to e the second converges to 1, so the limit is again e.

Similarly for (c), the sequence $\left(1+\frac{1}{n}\right)^{n+1} = \left(1+\frac{1}{n}\right)^{n} \cdot \left(1+\frac{1}{n}\right)$ converges to e.

For the limit in (d) $\left(1+\frac{1}{n^2}\right)^{n} = \left(\left(1+\frac{1}{n^2}\right)^{n^2}\right)^{\frac{1}{n}}$ we use that $2 \leq \left(1+\frac{1}{n}\right)^{n} \leq 4$. Therefore, $2^{\frac{1}{n}} \leq \left(1+\frac{1}{n^2}\right)^{n} \leq 4^{\frac{1}{n}}$. By theorem 16.4 both the upper and lower bounds converge to 1, thus by the squeeze theorem, $\lim_{n\to\infty}\left(1+\frac{1}{n^2}\right)^{n} = 1$.

Similarly, for (e), we get $2^n \leq \left(1+\frac{1}{n}\right)^{n^2}$, therefore $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n^2} = \infty$.

For (f) we simply note that $\left(1+\frac{1}{2n}\right)^{2n} = \left(\left(1+\frac{1}{2n}\right)^{2n}\right)^{\frac{1}{2}}$. The sequence $\left(1+\frac{1}{2n}\right)^{2n}$ converges to e, therefore by Exercise 12.6, we also have $\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^{2n} = e^{\frac{1}{2}}$.

Exercise 16.6 For part (a), we fix $0 \leq a < b$. We show $\frac{b^{n+1}-a^{n+1}}{b-a} > (n+1)a^n$ for $n \in \mathbb{N}$. Simply note that $(b-a)(b^n + b^{n-1}a + \cdots + ba^{n-1} + a^n) = b^{n+1} - a^{n+1}$. Therefore $\frac{b^{n+1}-a^{n+1}}{b-a} = b^n + b^{n-1}a + \cdots + ba^{n-1} + a^n > (n+1)a^n$, where we used that $a < b$.

Setting $a = 1 + \frac{1}{n+1}$ and $b = 1 + \frac{1}{n}$, we have $a < b$ thus by (a)

$$\left(1+\frac{1}{n}\right)^{n+1} = b^{b+1} > a^{n+1} + (b-a)(n+1)a^n = a^n(a + (b-a)(n+1)) = \left(1+\frac{1}{n+1}\right)^{n}\left(1+\frac{1}{n+1}+\frac{1}{n}\right).$$

For part (c), we derive that $1 + \frac{1}{n+1} + \frac{1}{n} > (1+\frac{1}{n+1})^2 = 1 + \frac{2}{n+1} + \frac{1}{(n+1)^2}$. Since $n^2 + 2n + 1 > n^2 + 2n$, we get $(n+1)^2 > n(n+2)$, so $\frac{n+1}{n} > \frac{n+2}{n+1} = 1 + \frac{1}{n+1}$. Dividing by $n+1$ gives $\frac{1}{n} > \frac{1}{n+1} + \frac{1}{(n+1)^2}$ and this last inequality in fact implies $1 + \frac{1}{n+1} + \frac{1}{n} > 1 + \frac{2}{n+1} + \frac{1}{(n+1)^2}$.

Parts (b) and (c) combined give $\left(1+\frac{1}{n}\right)^{n+1} > \left(1+\frac{1}{n+1}\right)^{n+2}$, therefore the sequence $a_n = \left(1+\frac{1}{n}\right)^{n+1}$ is decreasing. From 16.5 (c), we know that the limit is e. We know that $e < a_n$ for any $n \in \mathbb{N}$. Setting $n = 2$, we get $e < a_2 = \frac{27}{8} < 3$.

Exercise 16.7 We show that every convergent sequence has a monotone subsequence. (In fact, it holds even that every real sequence has a monotone subsequence which can be concluded using the Bolzano-Weierstrass theorem.)

Let $a_n \to L \in \mathbb{R}$. We say that a statement holds for almost all of the elements of $\{a_n\}$ if it holds for all but finitely many elements. If $a_n = L$ for almost all $n \in \mathbb{N}$, then we have a constant subsequence. If this is not true, then $a_n > L$ or $a_n < L$ holds for infinitely many n. Assume without loss of generality that $a_n > L$ for infinitely many n, otherwise consider $b_n = 2L - a_n$. This means that there is a subsequence $a_{f(n)} > L$. We construct a monotone subsequence $\{a_{f\circ g(n)}\}$ of $\{a_{f(n)}\}$. Set $g(1) = 1$. Set $g(n+1) = \min\{m \in \mathbb{N} \mid m > g(n), a_{f(m)} < a_{f(g(n))}\}$. Note that the set in the definition above is not empty since $a_{f(m)} \to L$. The sequence $\{a_{f\circ g(n)}\}$ is a monotone decreasing subsequence of $\{a_n\}$.

Exercise 16.8 The nested interval theorem is an important theorem that has generalization in topological spaces, called the Cantor's intersection theorem.

Let a sequence of closed intervals $[a_n, b_n]$ be given, where $a_n \leq b_n$ such that the sequence decreases i.e. $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$. We need to show that the intersection is not empty, i.e. $\cap_{n\in\mathbb{N}}[a_n, b_n] \neq \emptyset$. Furthermore, $\lim_{n\to\infty}\{b_n - a_n\} = 0$ is a necessary and sufficient condition for $\cap_{n\in\mathbb{N}}[a_n, b_n]$ containing exactly one point.

Note that $\{a_n\}$ is a non-decreasing sequence bounded above by b_1, therefore $\lim_{n\to\infty} a_n = c$ exists. Fix $m \in \mathbb{N}$ and note that $a_n \leq b_m$ for all $n \geq m$, therefore by the squeeze theorem, $c \leq b_m$ for all $m \in \mathbb{N}$. By the monotonicity of $\{a_n\}$, we conclude that $a_m \leq c$ for all $m \in \mathbb{N}$. Combining these conclusions, we get $a_m \leq c \leq b_m$ for all $m \in \mathbb{N}$. This means that $c \in [a_m, b_m]$ for all $m \in \mathbb{N}$. Therefore also $c \in \cap_{m\in\mathbb{N}}[a_m, b_m]$.

Now assume in addition that $\lim_{n\to\infty} b_n - a_n = 0$. Assume that c_1, c_2 are distinct elements in $\cap_{m\in\mathbb{N}}[a_m, b_m]$. Choosing m so big such that $b_m - a_m < |c_1 - c_2|$, we get a contradiction to $\{c_1, c_2\} \subset [a_m, b_m]$.

On the other hand, if $\lim_{n\to\infty} b_n - a_n \neq 0$, there is still a limit $\lim_{n\to\infty} b_n - a_n = d > 0$ since the sequence $\{b_n - a_n\}$ is positive and non-increasing. We know that $\lim_{n\to\infty} a_n = c_1$ is in $\cap_{m\in\mathbb{N}}[a_m, b_m]$. Similarly $\lim_{n\to\infty} b_n = c_2$ exists and is an element of $\cap_{m\in\mathbb{N}}[a_m, b_m]$. Then $c_2 - c_1 = \lim_{n\to\infty} b_n - \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n - c_n = d > 0$, so $c_1 \neq c_2$.

Exercise 16.10 Assume that $0 < x \leq y^2$ and $a_0 = y$ and $a_n = \frac{x/a_{n-1} + a_{n-1}}{2}$. If $x = y^2$, then $a_n = y = \sqrt{x}$ for all n.

Let us first show the famous inequality between the geometric and arithmetic means (AM-GM inequality) of positive real numbers x and y: $\sqrt{xy} \leq \frac{x+y}{2}$. Equality holds iff $x = y$. This easily follows by expanding $0 \leq (x-y)^2 = x^2 + y^2 - 2xy$, therefore $4xy \leq x^2 + y^2 + 2xy = (x+y)^2$. Dividing by 4 and taking the square root yields the inequality.

Assume that $x < y^2$. Note $a_n > 0$ for all n by induction over n. Using the AM-GM inequality directly for x/a_{n-1} and a_{n-1}, we conclude $\sqrt{x} \leq a_n$, thus $x \leq a_n^2$ for all n. However, if we also use that equality holds if and only if $x/a_{n-1} = a_{n-1}$, induction over n shows that $x < a_n^2$ for all n.

The sequence $\{a_n\}$ is decreasing since $a_n = \frac{x/a_{n-1} + a_{n-1}}{2} = a_{n-1}\frac{x/(a_{n-1})^2 + 1}{2} < a_{n-1}$ which follows by $x < a_{n-1}^2$. The sequence $\{a_n\}$ is decreasing and bounded below by 0, therefore it has a limit L. Taking the limit on both sides of the equation $a_n = \frac{x/a_{n-1} + a_{n-1}}{2}$, we get $L = \frac{x/L + L}{2}$, so $x = L^2$, i.e. $L = \sqrt{x}$.

For the case that $0 < y^2 < x$, we can conclude by the AM-GM inequality that $\sqrt{x} < a_1$, so we are back to the previously discussed case.

Similarly, for $k > 2$, $0 < x, y$ and $a_0 = y$ the sequence $a_n = \frac{x/(a_{n-1})^{k-1} + (k-1)a_{n-1}}{k}$ converges to $x^{\frac{1}{k}}$. The strategy of the proof is the same, except that we use the generalized AM-GM inequality $\sqrt[k]{x_1 \cdots x_k} \leq \frac{x_1 + \cdots + x_k}{k}$ with equality iff $x_1 = x_2 = \cdots = x_k$.

Remark: This method is very general and called Newton's method for finding zeros of functions. In these cases, we applied Newton's method to the function $f(y) = y^k - x$. However, at this point of the course we lack the techniques to introduce the method in its general form, but you can look it up somewhere if you are interested.

Exercise 16.14 If $a_n > 0$ and $\frac{a_{n+1}}{a_n} < 1$, then $\{a_n\}$ converges because it is bounded below by 0 and decreasing, since $a_{n+1} < a_n$.

For part (b), assume that $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = c < 1$. The proof is basically the same as for exercise 12.7 from homework 4.

Fix $\gamma > 0$ such that $c < \gamma < 1$. We can pick $N \in \mathbb{N}$ such that for all $n \geq N$, we have $\frac{a_{n+1}}{a_n} < \gamma$, i.e. $a_{n+1} < \gamma a_n$. The inequality $a_{n+1} < \gamma a_n$ holds for any $n \geq N$, therefore by iteration, for any $k \in \mathbb{N}$, we have

$$0 < a_{N+k} < \gamma^k a_N.$$

By theorem 16.3, we know that $\lim_{k\to\infty} \gamma^k = 0$ because $|\gamma| < 1$. Therefore $\lim_{k\to\infty} \gamma^k a_N = 0$ and by the squeeze theorem, we conclude that $\lim_{k\to\infty} a_{N+k} = 0$, therefore also $\lim_{n\to\infty} a_n = 0$.