Suggested solutions for Homework 6

Disclaimer: There might be some typos or mistakes, let me know if you find any. Therefore, double check! The solutions presented here are more concise than what is expected in quizzes and tests.

Notation: $\mathbb{N} = \{\text{positive integers}\}, \mathbb{N}_0 = \{\text{non-negative integers}\}, \mathbb{Z} = \{\text{integers}\}, \mathbb{Q} = \{\text{rational numbers}\}, \mathbb{R} = \{\text{real numbers}\}$

$\mathbb{Q}_{>0}$ stands for positive rationals, similarly for $\mathbb{Q}_{<0}, \mathbb{Q}_{\geq 0}, \mathbb{Q}_{\leq 0}$.

$X \approx Y$ means X and Y are equivalent (they have the same cardinality). For a finite set X, let $|X| \in \mathbb{N}_0$ be the number of elements of X.

Exercise 17.1 We show first (iv), (iii), (v) in this order. Note that by Exercise 7.5 we have these identities for integer valued x, now we need to extend them for $x \in \mathbb{R}$. First note that $a^x = \left(\frac{1}{a}\right)^{-x}$ holds for any $x \in \mathbb{R}$ and $a > 0$ (not just for $0 < a < 1$ as in Definition 17.2), to see this simply apply the definition to $\frac{1}{a}$, instead of a.

For (iv) note that $a^{-x}a^x = 1$, by (i), and therefore $a^{-x} = \frac{1}{a^x}$.

For (iii) fix $a, b > 1$ and $x \in \mathbb{R}$ and a sequence of rationals $r_n \nearrow x$. Then

$$(ab)^x = \lim_n (ab)^{r_n} = \lim_n a^{r_n} b^{r_n} = \lim_n a^{r_n} \lim_n b^{r_n} = a^x b^x.$$

If either $a = 1$, or $b = 1$, the statement is clear. For $0 < a, b < 1$, we have $ab < 1$, so $\frac{1}{ab} > 1$, therefore

$$(ab)^x = \left(\frac{1}{ab}\right)^{-x} = \left(\frac{1}{a}\right)^{-x} \left(\frac{1}{b}\right)^{-x} = a^x b^x.$$

For the case $a > 1$, $b < 1$ and $ab > 1$, fix a sequence of rationals $r_n \nearrow x$. Then

$$(ab)^x = \lim_n (ab)^{r_n} = \lim_n a^{r_n} b^{r_n} = \lim_n a^{r_n} \left(\frac{1}{b}\right)^{-r_n} = \lim_n a^{r_n} \lim_n \left(\frac{1}{b}\right)^{-r_n} = a^x \lim_n \left(\frac{1}{b}\right)^{-r_n},$$

and we only need to show that $\lim_n \left(\frac{1}{b}\right)^{-r_n} = b^x$, however $\left(\frac{1}{b}\right)^{-r_n} = \frac{1}{\left(\frac{1}{b}\right)^{r_n}} \to \frac{1}{\left(\frac{1}{b}\right)^{x}} = \frac{1}{b^{-x}}$. By (iv), we see that the latter equals b^x. Since a and b are interchangeable, we also obtain the case $a < 1$, $b > 1$ and $ab > 1$. It remains to show the case where $ab < 1$. Applying the pervious results to $\frac{1}{ab} > 1$, we get

$$(ab)^{-x} = \left(\frac{1}{ab}\right)^{x} = \left(\frac{1}{a}\right)^{x} \left(\frac{1}{b}\right)^{x} = a^{-x} b^{-x},$$

and thus $a^x b^x = (ab)^x$ by (i). This concludes the proof of (iii).

For (v), we have by (iii) in the second equality and (iv) in the one before the last equality

$$\left(\frac{a}{b}\right)^x = \left(a\frac{1}{b}\right)^x = a^x \left(\frac{1}{b}\right)^x = a^x b^{-x} = a^x \frac{1}{b^x} = \frac{a^x}{b^x}.$$

Let's prove (vi), (vii). We know that (vi) holds for $x, y \in \mathbb{Z}$ by Exercise 7.5 (g). Fix $x, y \in \mathbb{Q}$ and write them as $x = \frac{p_x}{q}$, $y = \frac{p_y}{q}$ for some $p_x < p_y \in \mathbb{Z}$ and $q \in \mathbb{N}$. We have $a^{p_x} < a^{p_y}$, therefore $(a^{p_x})^{\frac{1}{q}} < (a^{p_y})^{\frac{1}{q}}$, therefore $a^x < a^y$. Now fix $x < y \in \mathbb{R}$, and $p, q \in \mathbb{Q}$ such that $x < p < q < y$. There are rational sequences $r_n \nearrow x$

and $s_n \nearrow y$ such that $r_n < x$ and $q < s_n$, therefore $a^{r_n} < a^p < a^q < a^{s_n}$. Taking the limit in n, we obtain $a^x \leq a^p < a^q \leq a^y$, proving (vi). The inequality (vii) now follows from (vi) and (i).

Similarly for (viii) and (ix), we already have (viii) for $x \in \mathbb{N}$, this generalizes for $x \in \mathbb{Q}_{>0}$ as before. For $a > 1$ and $p \in \mathbb{Q}_{>0}$, we have $1 < a^p$. Fix $x > 0$ and positive, rational sequence $r_n \nearrow x$, then $1 < a^{r_1} \leq a^{r_n}$ and the right hand side converges to a^x, so $1 < a^x$. Now (viii) follows from (v). The inequality (ix) follows from (viii) and (i).

We left (ii), the hardest one to the end. We need to show that $(a^x)^y = a^{xy}$ for any $a > 0$ and $x, y \in \mathbb{R}$. We can conclude this identity for $x \in \mathbb{R}$ and $y \in \mathbb{Q}$ by (i) as follows. For positive y, write $y = \frac{p}{q}$ for $p, q \in \mathbb{N}$. Then by (i), $(a^x)^p = a^x \cdot a^x \cdots a^x = a^{x+x+\cdots+x} = a^{xp}$. Again by (i), we can verify that the q'th power of $a^{xp/q}$ is a^{xp}, therefore $(a^x)^y = a^{xy}$. For $y \in \mathbb{Q}_{<0}$ use the positive case and (i), for $y = 0$, the equality is obviously true.

For $a > 1$, $x > 0$ and $y \in \mathbb{R}$ arbitrary, fix an increasing, rational sequence $s_n \nearrow y$. Then $(a^x)^{s_n} = a^{xs_n}$, where the left hand side converges to $(a^x)^y$. We need to show that the right hand side converges to a^{xy}. This is not just the definition of real powers because xs_n might not be rational. However, there are rational, increasing sequences $r_n \nearrow xy$ and $r'_n \nearrow xy$ such that $r_n \leq xs_n \leq r'_n$. Then by (vi), we have $a^{r_n} \leq a^{xs_n} \leq a^{r'_n}$, where both the upper and lower bounds converge to a^{xy}, therefore by the squeeze theorem $a^{xs_n} \to a^{xy}$. This proves (ii) for the case of $a > 1$ and $x > 0$ and $y \in \mathbb{R}$ arbitrary.

For the case $x > 0$ and $0 < a < 1$, we can use (v) in the second, the previous case in the third and (iv) and (v) in the fourth step to conclude

$$(a^x)^{-y} = \left(\frac{1}{a^x}\right)^y = \left(\left(\frac{1}{a}\right)^x\right)^y = \left(\frac{1}{a}\right)^{xy} = a^{-xy}.$$

Using (i), now we conclude $(a^x)^y = a^{xy}$ in this case.

We are left with the case of negative x. So fix $x < 0$ and $a > 0$ and $y \in \mathbb{R}$ arbitrary. Then using the previous cases in the second step, we conclude

$$(a^x)^y = \left(\left(\frac{1}{a}\right)^{-x}\right)^y = \left(\frac{1}{a}\right)^{-xy} = a^{xy}.$$

Note that in the cases $a = 1$ or $x = 0$, both sides are 1, so the identity also holds true in these cases. Now, we covered all the cases, so we finished the proof of (ii).

Exercise 17.2 Let $x \in \mathbb{R}$ and r_n be a rational sequence such that $r_n \searrow x$. Let $a > 1$, then by Theorem 17.4 (vi), we know that a^{r_n} is decreasing and bounded below by a^x. Therefore $L = \lim_{n\to\infty} a^{r_n}$ exists and $a^x \leq L$. Without loss of generality (Theorem 17.4 (i)) assume that $x = 0$ by replacing the sequence r_n by $s_n = r_n - x \searrow 0$. We want to show that $L = 1$. Clearly $0 \leq L \leq a^{s_n}$. Since $s_n \searrow 0$, for each $n \in \mathbb{N}$, there exists an $m(n) \in \mathbb{N}$ such that $s_n \leq \frac{1}{m(n)}$. Choosing $m(n)$ to be maximal with that property, we see that $m(n)$ is non-decreasing (because s_n is decreasing) and converges to infinity (because s_n converges to 0). Therefore $1 \leq L \leq a^{s_n} \leq a^{\frac{1}{m(n)}}$. Since $m(n) \nearrow \infty$ as $n \to \infty$, we see that $a^{\frac{1}{m(n)}} \to 1$, by Theorem 16.4. By the squeeze theorem, we conclude that $L = 1$ proving the exercise for the case $a > 1$. For $a = 1$, the statement is clear. For $0 < a < 1$, we we see that $a^{r_n} = \left(\frac{1}{a}\right)^{-r_n} = \frac{1}{\left(\frac{1}{a}\right)^{r_n}}$ converges to $\frac{1}{\left(\frac{1}{a}\right)^x} = a^x$.

Exercise 17.3 Similarly, as in the last sentence of the solution to Exercise 17.2, we can show that $a^{r_n} \to a^x$ if for the rational sequence r_n, we have $r_n \nearrow x$ and $0 < a < 1$. In conclusion, for any monotone, rational sequence $r_n \to x$ and $a > 0$, we have $\lim_n a^{r_n} = a^x$. Taking an arbitrary rational sequence $r_n \to x$, there exists increasing, rational $s_n \nearrow x$ and decreasing, rational $s'_n \searrow x$ such that $s_n \leq r_n \leq s'_n$. Then in the case $a > 1$ we have by theorem 17.4 that $a^{s_n} \leq a^{r_n} \leq a^{s'_n}$. Both the upper and lower bounds converge to a^x, so by the squeeze theorem, we have $\lim_n a^{r_n} = a^x$. Similarly, for $0 < a < 1$, we have $a^{s'_n} \leq a^{r_n} \leq a^{s_n}$ and the conclusion $\lim_n a^{r_n} = a^x$ follows by the squeeze theorem.

Exercise 18.1 The sequence $a_n = n$ has no convergent subsequence, since for any subsequence $\{a_{n_k}\}$, we have $a_{n_k} \geq k$, thus diverges to $+\infty$.

Exercise 18.2 In Exercise 16.7 from homework 5, we showed that every convergent sequence has a monotone subsequence.

Let us distinguish the cases $\{a_n\}$ is bounded and $\{a_n\}$ is not bounded. If $\{a_n\}$ is bounded, it has a convergent subsequence $\{a_{n_k}\}$ by Bolzano-Weierstrass. By Exercise 16.7, $\{a_{n_k}\}$ has monotone subsequence, which is a monotone subsequence of $\{a_n\}$.

If $\{a_n\}$ not bounded, than it is either not bounded below, or not bounded above. Assume that $\{a_n\}$ is not bounded above. Then we can construct a monotone increasing subsequence $\{b_n\}$ as follows. Set $b_1 = a_1$. Since $\{a_n\}$ is not bounded above by a_1, there is a $n_2 \in \mathbb{N}$ such that $a_{n_2} > a_1$. Let $b_2 = a_{n_2}$. Assume that the $b_1 < b_2 < \cdots < b_m = a_{n_m}$ are already constructed. But since $\{a_n\}$ is not bounded by $M = \max(a_1, a_2, \cdots, a_{n_m})$, there exists an $n_{m+1} \in \mathbb{N}$ such that $a_{n_{m+1}} > M \geq b_m$. Now set $b_{m+1} = a_{n_{m+1}}$.

If $\{a_n\}$ is not bounded below, then $\{-a_n\}$ is not bounded above and it has an increasing subsequence $\{b_n\}$. Then $\{-b_n\}$ is a decreasing subsequence of $\{a_n\}$.

Exercise 18.3 Assume that every convergent subsequence of the bounded sequence $\{a_n\}$ converges to L. We need to show that $\{a_n\}$ converges to L. By Bolzano-Weierstrass, $\{a_n\}$ has a convergent subsequence $a_{n_k} \to L$. Now assume that $a_n \not\to L$. Then there exists an $\epsilon > 0$ and infinitely many $n \in \mathbb{N}$ such that $|a_n - L| \geq \epsilon$. Then there are infinitely many n such that $L + \epsilon \leq a_n$, or there are infinitely many n such that $a_n \leq L - \epsilon$. Assume that the former is true (the other case is similar). From those infinitely many $L + \epsilon \leq a_n$, we can build a subsequence $\{b_n\}$ of $\{a_n\}$ such that $L + \epsilon \leq b_n$. Note that $\{b_n\}$ is bounded, since it is a subsequence of $\{a_n\}$. Now, by Bolzano-Weierstrass, $\{b_n\}$ has a convergent subsequence $\{b_{n_j}\}$. However, since $L + \epsilon \leq b_{n_j}$, we have that $L + \epsilon \leq \lim_{j\to\infty} b_{n_j}$. Note that $\{b_{n_j}\}$ is a convergent subsequence of $\{a_n\}$ with limit different from L. This is contradiction, and therefore $\{a_n\}$ converges to L.