Suggested solutions for Homework 7

Disclaimer: There might be some typos or mistakes, let me know if you find any. Therefore, double check! The solutions presented here are more concise than what is expected in quizzes and tests.

Notation: $\mathbb{N} = \{\text{positive integers}\}, \mathbb{N}_0 = \{\text{non-negative integers}\}, \mathbb{Z} = \{\text{integers}\}, \mathbb{Q} = \{\text{rational numbers}\}, \mathbb{R} = \{\text{real numbers}\}$

$\mathbb{Q}_{>0}$ stands for positive rationals, similarly for $\mathbb{Q}_{<0}, \mathbb{Q}_{\geq 0}, \mathbb{Q}_{\leq 0}$.

$X \approx Y$ means X and Y are equivalent (they have the same cardinality). For a finite set X, let $|X| \in \mathbb{N}_0$ be the number of elements of X.

Exercise 19.1 Let a_n, b_n be Cauchy. Fix $\epsilon > 0$. We have $|a_n + b_n - (a_m + b_m)| = |(a_n - a_m) + (b_n - b_m)| \leq |a_n - a_m| + |b_n - b_m|$. Choosing N so big such that for all $n, m \geq N$, we have $|a_n - a_m| < \epsilon/2$ and $|b_n - b_m| < \epsilon/2$, we also have $|a_n + b_n - (a_m + b_m)| \leq \epsilon$. Hence $\{a_n + b_n\}$ is also Cauchy.

For $c \neq 0$, we choose N so big such that $|a_n - a_m| < \epsilon/|c|$ for all $n, m \geq N$, gives that $|ca_n - ca_m| = |c||a_n - a_m| < \epsilon$ for all $n, m \geq N$. Consequently, $\{ca_n\}$ is Cauchy. (For $c = 0$ this is obvious.)

For $\{a_n b_n\}$ note that $|a_n| \leq A$ and $|b_n| \leq B$ are bounded since they are Cauchy. Now writing $|a_n b_n - a_m b_m| = |a_n b_n - a_m b_n + a_m b_n - a_m b_m| \leq |a_n b_n - a_m b_n| + |a_m b_n - a_m b_m| = |b_n|\ |a_n - a_m| + |a_m|\ |b_n - b_m| \leq B\ |a_n - a_m| + A\ |b_n - b_m|$. Choosing N so big such that for all $n, m \geq N$ we have $|a_n - a_m| < \frac{\epsilon}{2B}$ and $|b_n - b_m| < \frac{\epsilon}{2A}$, we also have $|a_n b_n - a_m b_m| < \epsilon$. Consequently, $\{a_n b_n\}$ is Cauchy.

The sequence $\{a_n - b_n\}$ is Cauchy because $\{-b_n\}$ is Cauchy and therefore $\{a_n + (-b_n)\}$ is also Cauchy.

Exercise 19.3 We can conclude that $|a_3 - a_2| = |f(a_2) - f(a_1)| \leq \alpha|a_2 - a_1|$. Also $|a_4 - a_3| = |f(a_3) - f(a_2)| \leq \alpha|a_3 - a_2|$. Combining these inequalities, we have $|a_4 - a_3| \leq \alpha^2|a_2 - a_1|$. Proceeding like this, we see that $|a_{n+1} - a_n| \leq \alpha^{n-1}|a_2 - a_1|$. Denote $d = |a_2 - a_1|$ and assume $d \neq 0$ (otherwise the statement is obviously true) also fix positive integers $n < m$. We have the following telescoping sum $a_m - a_n = \sum_{k=0}^{m-n-1} a_{n+k+1} - a_{n+k}$, and therefore by the triangular inequality in the first step, the conclusion from above in the second, and the identity $\sum_{k=0}^{n} \gamma^k = \frac{1-\gamma^{n+1}}{1-\gamma}$ for $\gamma \neq 0$ in the third, we get

$$|a_m - a_n| \leq \sum_{k=0}^{m-n-1} |a_{n+k+1} - a_{n+k}| \leq d \sum_{k=0}^{m-n-1} \alpha^{n+k-1} = d\alpha^{n-1} \frac{1 - \alpha^{m-n}}{1 - \alpha} \leq \alpha^{n-1} d \frac{1}{1 - \alpha}.$$

In the last step, we used that the sequence $\left\{\frac{1-\alpha^{m-n}}{1-\alpha}\right\}_m$ is increasing with limit $\frac{1}{1-\alpha}$, that is true, since $\alpha \in [0, 1)$. Fix $\epsilon > 0$. For the same reason as before $\lim_n \alpha^{n-1} = 0$, and therefore we can take N to be so big such that $\alpha^{n-1} < \frac{1-\alpha}{d}\epsilon$ for all $n \geq N$. This means that for any $m > n \geq N$, we have $|a_m - a_n| < \epsilon$. But then it should also hold for arbitrary $m, n \geq N$, which proves that $\{a_n\}$ is a Cauchy sequence.

Exercise 20.6 Since $\frac{1}{n} \to 0$, we have $\limsup \frac{1}{n} = \liminf \frac{1}{n} = 0$. Similarly $\left(1 + \frac{1}{n}\right)^n \to e$, therefore the limes superior equals the limes inferior equals the limit e.

For the sequence $a_n = (-1)^n(1 - \frac{1}{n})$ note that $1 - \frac{1}{n} \to 1$. Since a convergent subsequence a_{n_k} is Cauchy, either all but finitely many of it's elements are of the form a_{2n} or all but finitely many of it's elements are of the form a_{2n-1}. In the first case $\lim_k a_{n_k} = \lim_n a_{2n} = 1$, in the second $\lim_k a_{n_k} = \lim_n a_{2n-1} = -1$. This means that the set of accumulations points $\mathcal{L}_a = \{1, -1\}$. Therefore $\liminf_n a_n = -1$ and $\limsup_n a_n = 1$.

Exercise 20.7 Let a_n be an enumeration of $[0, 1] \cap \mathbb{Q}$. This is to say that $a : \mathbb{N} \to [0, 1] \cap \mathbb{Q},\ n \mapsto a_n$ is a bijection. We have $0 \leq a_n \leq 1$. Therefore, $0 \leq \liminf_n a_n$ and $\limsup_n a_n \leq 1$. We show that these inequalities are in fact equalities. We only show $\limsup_n a_n = 1$, the limes inferior is similar. Assume $L = \limsup_n a_n < 1$. By Theorem 20.3, there are at most finitely many n such that $a_n \geq L + \frac{1-L}{2} = \frac{L+1}{2}$. However there are infinitely many rational numbers in the interval $\left[\frac{L+1}{2}, 1\right]$, which contradicts that a is onto $[0,1] \cap \mathbb{Q}$.

Exercise 20.8 We need to show that $\{\sup \mathcal{L}_a, \inf \mathcal{L}_a\} \subset \mathcal{L}_a$, i.e. the set $\mathcal{L}_a$ contains both its supremum and infimum. We only show it for $\sup \mathcal{L}_a$, the infimum is similar. By Theorem 20.3 we know that for any $\epsilon > 0$ there are infinitely many n such that $L - \epsilon < a_n$. Therefore we can build a subsequence $\{b_k\}$ of $\{a_n\}$ such that $L - \frac{1}{k} < b_k$ as follows. Set b_1 to be such that $L - 1 < b_1$. Now assume that $b_1, \ldots, b_k = a_{n_k}$ are already constructed. Since there are infinitely n such that $L - \frac{1}{k+1} < a_n$, choose n_{k+1} to be such an n which is greater than n_k, i.e. $n_k < n_{k+1}$. Now set $b_k = a_{n_k}$.

Fix $\epsilon > 0$ and $K \in \mathbb{N}$ such that $\frac{1}{K} < \epsilon$. The sequence $\{b_k\}$ has the property that for any $k \geq K$, we have that $L - \frac{1}{K} < b_k$. However, by Theorem 20.3 there exists a N such that $a_n < L + \frac{1}{K}$ for all $n \geq N$. This also means that there exists a K' such that for all $k \geq K'$, we have $b_k < L + \frac{1}{K}$. Now setting $K'' = \max\{K, K'\}$, we have that $L - \frac{1}{K} < b_k < L + \frac{1}{K}$ for all $k \geq K''$. Therefore $|L - b_k| < \epsilon$ for all $k \geq K''$, in other words $\lim_k b_k = L$. In conclusion $L \in \mathcal{L}_a$, this is what we wanted to show.

Exercise 20.9 This is the same as **Exercise 18.3** form homework 6. We can give a shorter proof using limes inferior, limes superior. By Bolzano-Weierstrass and the assumption, we have $\mathcal{L}_a = \{L\}$. Therefore, $\liminf_n a_n = \limsup_n a_n = L$. By Theorem 20.4 (ii), the sequence $\{a_n\}$ converges and has limit L.

Exercise 20.10 The sequence $a_n = (-1)^n$ diverges but $\frac{a_1 + \cdots + a_n}{n}$ converges to zero. This is because the sequence $b_n = a_1 + a_2 + \cdots + a_n$ is bounded by 1, i.e. $|b_n| \leq 1$. Therefore $\left|\frac{a_1 + \cdots + a_n}{n}\right| \leq \frac{1}{n}$. By the squeeze theorem the sequence $\left|\frac{a_1 + \cdots + a_n}{n}\right|$ converges to 0, hence $\frac{a_1 + \cdots + a_n}{n}$ also converges to 0.

Exercise 20.20 Let $0 < a_n \to L$, we need to prove that $(a_1 \cdots a_n)^{\frac{1}{n}} \to L$.

First we show that for any $a > 0$ there is a unique real r such that $2^r = a$. We will call this r the base-2 logarithm of a and write $r = \log_2 a$.

Let $r = \sup\{x \in \mathbb{R} : 2^x \leq a\}$, with the set $S = \{x \in \mathbb{R} : 2^x \leq a\}$ which is not empty since $2^{-n} = \frac{1}{2^n} \to 0$. By Exercise 17.3 from homework 6, we have that $2^r \leq a$. Assume that $2^r < a$. For $n \in \mathbb{N}$, we have by Theorem 17.4 (i) that $2^{r+\frac{1}{n}} = 2^r\ 2^{\frac{1}{n}}$. Not that $1 < 2^{\frac{1}{n}} \to 1$ by Theorem 16.4.. Choosing n so big such that $2^{\frac{1}{n}} < \frac{a}{2^r}$, we obtain that $2^{r+\frac{1}{n}} = 2^r 2^{\frac{1}{n}} < a$ which is contradiction. Uniqueness follows form Theorem 17.4 (vi).

From Theorem 17.4 (i) we have for $a, b > 0$ that $\log_2(ab) = \log_2 a + \log_2 b$ and from (ii) is easy to check that for $a > 0$ and $x \in \mathbb{R}$, we have $\log_2 a^x = x \log_2 a$.

Let us show that $a_n \to L$ if and only if $\log_2 a_n \to \log_2 L$. If $\log_2 a_n \to \log_2 L$, then we know by Exercise 17.3 that $2^{\log_2 a_n} \to 2^{\log_2 L}$ which is the same as $a_n \to L$. Now assume that $b_n = \log_2 a_n$ doesn't converge to $M = \log_2 L$. Then (as in the solution of Exercise 18.3) there exists $\epsilon > 0$ such that there are infinitely many n such that $b_n \geq M + \epsilon$ or there are infinitely many n such that $b_n \leq M - \epsilon$. In the first case $a_n = 2^{b_n} \geq 2^M 2^\epsilon = L2^\epsilon$ for infinitely many n which contradicts $a_n \to L$. Similarly, for the second case $a_n = 2^{b_n} \leq 2^M 2^{-\epsilon} = L2^{-\epsilon}$ for infinitely many n which again contradicts $a_n \to L$.

Now we show that $\log_2(a_1 \cdots a_n)^{\frac{1}{n}} \to \log_2 L$. Using the identities above $\log_2(a_1 \cdots a_n)^{\frac{1}{n}} = \frac{\log_2 a_1 + \cdots + \log_2 a_n}{n}$. Using Theorem 20.7 and that $\log_2 a_n \to \log_2 L$, we have that $\log_2(a_1 \cdots a_n)^{\frac{1}{n}} \to \log_2 L$. As we showed before, this implies that $(a_1 \cdots a_n)^{\frac{1}{n}} \to L$.

Exercise 20.22 Let $b_n \to L$. We show that $\frac{a_{2n}}{2n}$ and $\frac{a_{2n+1}}{2n+1}$ both converge to zero that implies $\frac{a_n}{n} \to 0$.

We build the sum $b_2 + \cdots + b_{n+1} = a_1 + 2(a_2 + \cdots + a_n) + a_{n+1}$. This means that $a_{n+1} = b_2 + \cdots + b_{n+1} - a_1 - 2(a_2 + \ldots a_n)$. Dividing by n gives

$$\frac{a_{n+1}}{n} = \frac{b_2 + \cdots + b_{n+1}}{n} - 2\frac{a_2 + \cdots + a_n}{n} - \frac{a_1}{n}.$$

We know by Theorem 20.7 that $\frac{b_2 + \cdots + b_{n+1}}{n} \to L$ and we also know that $\frac{a_1}{n} \to 0$. If n is odd, then $a_2 + \cdots + a_n =$

$b_3 + b_5 + \cdots + b_n$ and by Theorem 20.7 we know that

$$\frac{b_3 + b_5 + \cdots + b_n}{(n-1)/2} \to L,$$

therefore also $2\frac{a_2+\cdots+a_n}{n} = 2\frac{b_3+b_5+\cdots+b_n}{n} = \frac{n-1}{n}\frac{b_3+b_5+\cdots+b_n}{(n-1)/2} \to L$ proving that $\frac{a_{2m}}{2m-1} \to 0$ (here the even number $n+1 = 2m$) which in turn implies that $\frac{a_{2m}}{2m} = \frac{2m-1}{2m}\frac{a_{2m}}{2m-1} \to 0$.

Now we consider the shifted sequence $\{a_{n+1}\}$. Defining $\tilde{b}_{n+1} = a_{n+1} + a_{n+2}$, we know that $\tilde{b}_n \to L$. However, the derivation above gives that $\frac{a_{2n+1}}{2n} \to 0$, which implies $\frac{a_{2n+1}}{2n+1} \to 0$.

Exercise 21.2 For (a), $a_n = (-1)^n$ and we have $A_n = 1$, and $B_n = -1$. For (b), $a_n = \frac{1}{n}$ and we have $A_n = \frac{1}{n} \to 0$ and $B_n = 0$. For (c), $a_n = \left(1 + \frac{1}{n}\right)^n$ and $A_n = e$ and $B_n = \left(1 + \frac{1}{n}\right)^n$.

For (d), $a_n = \frac{(-1)^n}{n}$ and $A_{2n} = \frac{1}{2n}$, $A_{2n+1} = \frac{1}{2n+2}$, $B_{2n+1} = \frac{-1}{2n+1}$, $B_{2n} = \frac{-1}{2n+1}$. Therefore $\lim_n A_n = \lim_n B_n = 0$.

For (e), we have $a_n = (-1)^n\left(1 - \frac{1}{n}\right)$. Note that $\{a_{2n}\}$ is increasing and positive, and $\{a_{2n-1}\}$ is decreasing and non positive. Therefore $A_n = \sup\{a_{2m} | m \in \mathbb{N}, 2m \geq n\} = \lim_n a_{2n} = 1$ and $B_n = \inf\{a_{2m-1} | m \in \mathbb{N}, 2m-1 \geq n\} = \lim_n a_{2n-1} = -1$.