# Suggested solutions for Homework 8

Disclaimer: There might be some typos or mistakes, let me know if you find any. Therefore, double check! The solutions presented here are more concise than what is expected in quizzes and tests.

**Notation**: $\mathbb{N} = \{\text{positive integers}\}, \mathbb{N}_0 = \{\text{non-negative integers}\}, \mathbb{Z} = \{\text{integers}\}, \mathbb{Q} = \{\text{rational numbers}\}, \mathbb{R} = \{\text{real numbers}\}$ $\mathbb{Q}_{>0}$ stands for positive rationals, similarly for $\mathbb{Q}_{<0}, \mathbb{Q}_{\geq 0}, \mathbb{Q}_{\leq 0}$.

$X \approx Y$ means $X$ and $Y$ are equivalent (they have the same cardinality). For a finite set $X$, let $|X| \in \mathbb{N}_0$ be the number of elements of $X$.

**Exercise 22.1** The sequences $(-1)^n$ and $\frac{n^2}{n+1}$ are not convergent to 0. By theorem 22.3 the series $\sum_{n=1}^{\infty}(-1)^n$ and $\sum_{n=1}^{\infty} \frac{n^2}{n+1}$ are divergent.

**Exercise 22.4** Using that $a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$, we see that $s_n = a_1 + a_2 + \cdots + a_n = 1 - \frac{1}{n+1}$ is a telescoping sum. Since $s_n \to 1$, the series $\sum_{n=1}^{\infty} a_n$ is convergent with limit 1.

**Exercise 22.5** Using that $a_n = \frac{n}{(n+1)!} = \frac{n+1}{(n+1)!} - \frac{1}{(n+1)!} = \frac{1}{n!} - \frac{1}{(n+1)!}$, we see that $s_n = a_1 + a_2 + \cdots + a_n = 1 - \frac{1}{(n+1)!}$ is a telescoping sum with limit 1. Therefore the series $\sum_{n=1}^{\infty} a_n$ is convergent with limit 1.

**Exercise 22.7** Assume that $a_n = b_n$ for $n \geq N$ and $\sum_{n=1}^{\infty} a_n$ is convergent. Then we have for $n \geq N$ that $s_n = b_1 + b_2 + \cdots b_n = a_1 + a_2 + \cdots + a_n - (a_1 + a_2 + \cdots + a_{N-1}) + (b_1 + b_2 + \cdots + b_{N-1})$. Now, we see that as $n \to \infty$, the sequence $s_n$ converges to $\sum_{k=1}^{\infty} a_n - (a_1 + a_2 + \cdots + a_{N-1}) + (b_1 + b_2 + \cdots + b_{N-1}) = L - (a_1 + a_2 + \cdots + a_{N-1}) + (b_1 + b_2 + \cdots + b_{N-1})$. Therefore, $\sum_{k=1}^{\infty} b_n = L - (a_1 + a_2 + \cdots + a_{N-1}) + (b_1 + b_2 + \cdots + b_{N-1})$ is a convergent series.

**Exercise 23.4** Assume for the sake of contradiction that $\sum_{n=1}^{\infty} a_n + b_n$ converges. By theorem 23.1, also $\sum_{n=1}^{\infty} a_n + b_n - a_n$ converges. However the latter equals to $\sum_{n=1}^{\infty} b_n$ which doesn't converge, and this leads to a contradiction.

**Exercise 23.5** One example is $a_n = (-1)^n$, $b_n = (-1)^{n+1}$, then $a_n + b_n = 0$, therefore $\sum_{n=1}^{\infty} a_n + b_n$ converges to 0, however $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ diverge, since $a_n$ and $b_n$ does not converge to zero.

Another example is $a_n = n$, $b_n = -n$. Then $a_n + b_n = 0$, therefore $\sum_{n=1}^{\infty} a_n + b_n$ converges to 0, however $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ diverge, since $a_n$ and $b_n$ does not converge to zero.

**Exercise 24.2** In part (a), we apply the $2^n$ test, which says that $\sum_{n=2}^{\infty} \frac{1}{nL(n)}$ converges if and only if $\sum_{n=2}^{\infty} \frac{2^n}{2^n L(2^n)}$ converges. The latter simplifies to $\sum_{n=2}^{\infty} \frac{1}{nL(2)} = \frac{1}{L(2)} \sum_{n=2}^{\infty} \frac{1}{n}$ which diverges to $+\infty$ by the example in the book. The divergence of the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ is a very important fact.

For part (b) note that the $\frac{1}{nL(n)} \leq \frac{1}{L(n)}$ for all $n \geq 2$ and both sides are positive. The series $\sum_{n=2}^{\infty} \frac{1}{nL(n)}$ diverges to $+\infty$, by part (a), therefore $\sum_{n=2}^{\infty} \frac{1}{L(n)}$ diverges as well.

**Exercise 24.5** Since the sequence $a_n$ is non negative, for any $k$ we have that we $a_{n_1} + a_{n_2} + \cdots + a_{n_k} \leq a_1 + a_2 + \cdots + a_{n_k}$. The right hand side converges to some limit $0 \leq L$ as $k \to \infty$. Therefore $s_k = a_{n_1} + a_{n_2} + \cdots + a_{n_k} \leq L$ for any $k$. The sequence of partial sums $s_k$ is increasing and bounded from above by $L$. Therefore it is convergent with some limit $L' \leq L$.

**Exercise 24.7** Let $b_n > 0$ and $a_n \geq 0$ such that $\sum_{n=1}^{\infty} b_n$ converges and $\{\frac{a_n}{b_n}\}$ is decreasing. Then for any $n$, we have $\frac{a_n}{b_n} \leq \frac{a_1}{b_1}$, since $\{\frac{a_n}{b_n}\}$ is decreasing. Therefore, $a_n \leq \frac{a_1}{b_1} b_n$, where both sides of the inequality are non-negative. Therefore, the increasing sequence $s_n = a_1 + \cdot + a_n$ is bounded above by $\frac{a_1}{b_1} \sum_{n=1}^{\infty} b_n < \infty$, thus $s_n$ converges to some limit $L \leq \frac{a_1}{b_1} \sum_{n=1}^{\infty} b_n$.