Suggested solutions for Homework 9

Disclaimer: There might be some typos or mistakes, let me know if you find any. Therefore, double check! The solutions presented here are more concise than what is expected in quizzes and tests.

Notation: $\mathbb{N} = \{\text{positive integers}\}, \mathbb{N}_0 = \{\text{non-negative integers}\}, \mathbb{Z} = \{\text{integers}\}, \mathbb{Q} = \{\text{rational numbers}\}, \mathbb{R} = \{\text{real numbers}\}$ $\mathbb{Q}_{>0}$ stands for positive rationals, similarly for $\mathbb{Q}_{<0}, \mathbb{Q}_{\geq 0}, \mathbb{Q}_{\leq 0}$.

Exercise 25.1 Both $a_n = e-(1-1/n)^n$ and $b_n = n^{1/n}-1$ are decreasing sequences converging to zero, therefore convergence of the series follows from the alternating series test.

Exercise 25.2 $s_{2n+1} - s_{2n-1} = a_{2n+1} - a_{2n} \leq 0$. Also $s_{2n-1} = (a_1 - a_2) \pm ... + (a_{2n-3} - a_{2n-2}) + a_{2n-1} \geq a_{2n-1} \geq 0$.

Exercise 25.3 We show that for $m \geq n$, we have $|s_m - s_n| \leq a_{n+1}$. We distinguish between the cases n even or odd, we do the case when n is even. Let $m > n$ be both even, then

$$s_m - s_n = a_{n+1} - (a_{n+2} - a_{n+3}) - (a_{n+4} - a_{n+5}) - \cdots - (a_{m-2} - a_{m-1}) - a_m \leq a_{n+1}.$$

This inequality is also true if m is odd, then

$$s_m - s_n = a_{n+1} - (a_{n+2} - a_{n+3}) - (a_{n+4} - a_{n+5}) - \cdots - (a_{m-1} - a_m) \leq a_{n+1}.$$

Since $\{s_{2n}\}$ is increasing, we have for even m that $0 \leq s_m - s_n$. This holds also in the case that m is odd $0 \leq a_m + s_{m-1} - s_n = s_m - s_n$. Therefore, for any $m \geq n$, we have $|s_m - s_n| \leq a_{n+1}$. Letting $m \to \infty$, we get that $|L - s_n| \leq a_{n+1}$. The case of n odd is similar.

Exercise 25.4 We take the example $1 - 0 + 1/2 - 0 + 1/3 - 0 + 1/4 + \cdots$. We see that $a_n = 0$ if n is even, and $a_n = \frac{2}{n+1}$ if n is odd.

Exercise 26.1 We will use that $\sum_{n\geq 1} \frac{1}{n^s}$ converges for $s > 1$ and diverges for $s \leq 1$.

(a) $\frac{1}{n^3+n^2+n} \leq \frac{1}{n^2}$, therefore the series converges absolutely.

(b) By the alternating test, the series converges. However, $\frac{1}{n+\sqrt{n}} \geq \frac{1}{2n}$, so the series converges only conditionally.

(c) $\frac{\sqrt{n+1}-\sqrt{n}}{n} = \frac{1}{(\sqrt{n+1}+\sqrt{n})n} \leq \frac{1}{(\sqrt{n}+\sqrt{n})n} = \frac{1}{2n^{3/2}}$, so the series converges absolutely.

(d) $d_n = \frac{n}{2^n}$, calculating $d_{n+1}/d_n = \frac{n+1}{2n} \to \frac{1}{2}$, the series converges absolutely by the ratio test. Do (e) and (f) similarly to conclude absolute convergence.

(g) The sequence $\{n\ n^{(1/n)}\}$ is increasing, since $\frac{(n+1)(n+1)^{1/(n+1)}}{n\ n^{1/n}} \geq \frac{(n+1)\ n^{1/(n+1)}}{n\ n^{1/n}} = (1+1/n)\ n^{-1/(n(n+1))}$. We need to show that this is bigger than 1. Raising it to the power $n\ (n+1)$, we get $((1+1/n)^n)^{n+1} n^{-1} \geq 2^{n+1}\ n^{-1} \geq 2$, because $2^n \geq n$. The inequality $2^n \geq n$ can be for example be concluded from the binomial theorem $2^n = (1+1)^n = \sum_{k=0}^{n} \binom{n}{k} \geq n+1$, since $\binom{n}{k} \geq 1$. Therefore $\{n\ n^{(1/n)}\}$ is increasing, and the series converges by the alternating series test.

The series does not converge absolutely, since $n^{1+1/n} \leq 2n$ for n big enough. This holds since $n^{1/n} \to 1$, as $n \to \infty$.

(h) The series converges by the alternating series test, where we use that $n^{1/n} \to 1$ is a decreasing sequence. The sequence does not converge absolutely, since otherwise $n(n^{1/n} - 1) \to 1$ should hold (by problem 24.9 which follows using 2^n test). However, noting that $n^{1/n} \geq (1 + 1/n)$ as soon as $n \geq 4$, we have that $n(n^{1/n} - 1) \geq n(1 + 1/n - 1) = 1$ cannot converge to zero.

(i) The series converges by the alternating series test, where we use that $(1+1/n)^n \to e$ is an increasing sequence.

The series does not converge absolutely, since $e-(1+1/n)^n \geq 1/4n$ for $n \geq 2$. (This is somewhat weaker than what is claimed in the book, however satisfactory for us.)

A computation using the definition of factorials shows that for $k \geq 2$ (please don't learn this by heart)

$$\binom{2n}{k}\left(\frac{1}{2n}\right)^k - \binom{n}{k}\left(\frac{1}{n}\right)^k \geq \frac{1}{2n}\frac{k-1}{k}\binom{2n}{k-1}\left(\frac{1}{2n}\right)^{k-1} \geq \frac{1}{2n}\frac{k-1}{k(k-1)^{k-1}} = \frac{1}{2n}\frac{1}{k(k-1)^{k-2}},$$

where we used that $\binom{n}{k} \geq \frac{n^k}{k^k}$ in the second inequality. Note that the difference above is zero for $k=0,1$. By the binomial theorem, we have

$$\left(1+\frac{1}{n}\right)^n = \sum_{k=0}^{n}\binom{n}{k}\frac{1}{n^k}.$$

Since $(1+1/n)^n \to e$ is increasing, we obtain for $n \geq 2$

$$\begin{aligned} e-\left(1+\frac{1}{n}\right)^n &\geq \left(1+\frac{1}{2n}\right)^{2n} - \left(1+\frac{1}{n}\right)^n \\ &= \sum_{k=0}^{2n}\binom{2n}{k}\frac{1}{(2n)^k} - \sum_{k=0}^{n}\binom{n}{k}\frac{1}{n^k} \\ &\geq \sum_{k=0}^{n}\binom{2n}{k}\frac{1}{(2n)^k} - \binom{n}{k}\frac{1}{n^k} \geq \frac{1}{2n}\sum_{k=2}^{n}\frac{1}{k(k-1)^{k-2}} \geq \frac{1}{4n} \end{aligned}$$

(j) The series converges by the alternating series test. However, it does not converge absolutely, since $\frac{1\cdot 3 \ldots (2n-1)}{2\cdot 3 \ldots 2n} \geq \frac{1}{2n}$.

Exercise 26.4 Assume that $\sum_{n=1}^{\infty} a_n$ converges absolutely. Then $a_n \to 0$, therefore $|a_n| \leq 1$ for big enough n. This also means that $a_n^2 = |a_n|^2 \leq |a_n|$ for big enough n. Since $\sum_{n=1}^{\infty}|a_n|$ converges, also $\sum_{n=1}^{\infty} a_n^2$ converges.

26.8 $(a_n \geq 0)$ Let $\frac{a_{n+1}}{a_n} \to L$.

Fix $\varepsilon > 0$. We need to find N st.

$$|a_n^{1/n} - L| < \varepsilon \quad \text{for} \quad n \geq N.$$

Fix $\bar{N}$ st.

$$\left| \frac{a_{n+1}}{a_n} - L \right| < \varepsilon.$$

then for $n \geq \bar{N}$

$$(L-\varepsilon)^{n-\bar{N}} a_N \leq a_n = \frac{a_n}{a_{n-1}} \cdots \frac{a_{N+1}}{a_N} a_N$$

$$\leq (L+\varepsilon)^{n-N} a_N$$

so $$(L-\varepsilon)^{1-\frac{\bar{N}}{n}} |a_N|^{1/n} \leq |a_n|^{1/n} \leq (L+\varepsilon)^{1-\frac{\bar{N}}{n}} |a_N|^{\frac{1}{n}}$$

$$(L-\varepsilon) \underbrace{\left((L-\varepsilon)^{\bar{N}} a_N\right)^{1/n}}_{\substack{\to 1 \\ \text{as } n\to\infty}} \leq |a_n|^{1/n} \leq (L+\varepsilon) \underbrace{\left((L-\varepsilon)^{\bar{N}} a_N\right)^{\frac{1}{n}}}_{\substack{\to 1 \\ \text{as } n\to\infty}}$$

Fix $\delta > 0$.

Choose $N \geq 0$ s.t. for $n \geq N$: $\left((L-\varepsilon)^{N} a_N\right)^{1/n} \in (1-\delta, 1+\delta)$

so $$\underbrace{(L-\varepsilon)(1-\delta)}_{= L - \underbrace{(\delta L + \varepsilon(1-\delta))}_{\leq \bar{\varepsilon}}} \leq |a_n|^{1/n} \leq \underbrace{(L+\varepsilon)(1+\delta)}_{L + \underbrace{(\delta L + \varepsilon(1-\delta))}_{\leq \bar{\varepsilon}}}$$

Fix $\bar{\varepsilon} > 0$, choose Now δ, ε so small st $(\delta L + \varepsilon(1-\delta)) \leq \bar{\varepsilon}$.

Now for all $n \geq \max(N, \bar{N})$ we have

$\uparrow\uparrow$ depend on ε, δ (choice on $\bar{\varepsilon}$)

that

$$L - \bar{\varepsilon} \leq |a_n|^{1/n} \leq L + \bar{\varepsilon}.$$

Since $\bar{\varepsilon}$ was arbitrary, we showed that

$$|a_n|^{1/n} \to L.$$

27.1 Find R, the radius of convergence

a) $$\sum_{n=1}^{\infty} \underbrace{\frac{x^{2n-1}}{(2n-1)!}}_{= a_n}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|^{2n+1}}{(2n+1)!} \frac{(2n-1)!}{|x|^{2n-1}} = \frac{|x|^2}{(2n+1)\cdot 2n} \xrightarrow{n\to\infty} 0$$

therefore $R = \infty$.

b) $$\sum_{n=1}^{\infty} \frac{n\,(x-1)^n}{2^n}, \quad a_n := \frac{n}{2^n}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{2^{n+1}} \frac{2^n}{n} = \frac{1}{2}\frac{n+1}{n} \to \frac{1}{2}$$

so $R = 2$.

For $x = 1+2 = 3$ or $x = 1-2 = -1$: $\sum_{n=1}^{\infty} \frac{n}{2^n}(\pm 2)^n = \sum_{n=1}^{\infty} (\pm 1)^n n$ is divergent.

c) $$\sum_{n=1}^{\infty} \frac{n!}{2^n} (x+2)^n \quad , \quad a_n = \frac{n!}{2^n}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!}{2^n} \cdot \frac{2^{n+1}}{n!} = 2(n+1) \to \infty$$

So $R=0$.

d) $$\sum_{n=1}^{\infty} (n^{1/n}-1)\, x^n$$

summable for $x=-1$ (see 26.1)

not summable for $x=1$ — " —

so $\boxed{R=1}$

e) $$\sum_{n=2}^{\infty} \frac{(x-\pi)^n}{n(n-1)} \quad , \quad a_n := \frac{1}{n(n-1)}$$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n(n-1)}{(n+1)n} \to 1 \quad \text{so } R=1$$

For $x = \pi+1$: $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(n-1)}$ is convergent

because $n(n-1) = n^2 - n \geq \frac{n^2}{2}$ for n big.

and $\displaystyle\sum_{n=2}^{\infty} \frac{2}{n^2}$ is convergent.

For $x = \pi - 1$: $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n(n-1)}$ is convergent by the alternating series test.