Spontaneous breaking of rotational symmetry in a probabilistic hard disk model in Statistical Mechanics

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Contents

Abstract

Throughout this thesis, we are going to work in a hard disk model which is analogue to the setting in [MR09] and [HMR13]. In this model of two-dimensional crystals, every admissible configuration is a hard disk configuration, which, in addition, is a perturbed version of some triangular lattice with side length one. A triangular lattice with side length one is called standard. The objective of this thesis is to show that admissible configurations in a given box with side length $N \in \mathbb{N}$ are arbitrarily close to some standard triangular lattice whenever the particle density is chosen sufficiently high, and this choice can be made independent of the box size N. Since this result is independent of the choice of N, the existence of an infinite-volume Gibbs measure can be shown which, in spite of the rotational symmetry of the set of admissible configurations, breaks the rotational symmetry in a strong sense.

1 Motivation

The foundations of modern Statistical Mechanics were laid by Austrian physicist L. E. Boltzmann in the nineteenth century. The Boltzmann distribution describes a physical system in thermal equilibrium in terms of a given energy function, which is also called the Hamiltonian of the system. The probability density on the set of states is given by the Boltzmann factor and the partition sum. The Boltzmann factor carries information about the energy of different states; the partition sum is a normalising constant. Integrating the Boltzmann factor over a set of possible states, and subsequently dividing by the partition sum, one obtains the probability of the system to be in some state of the given set. A constraint of this model is the requirement that the partition sum must be finite. However, several experimental findings were able to be explained using the Boltzmann distribution such as the Maxwell-Boltzmann distribution of particles' speed in ideal gases or its refinements. The study of infinite systems was first made accessible when the notion of Gibbs measures was introduced in the 1960s. In the works [Dob68], [LR69] and [Dob70], mathematicians R. L. Dobrushin, O. E. Lanford and D. Ruelle introduced this new concept, which posed new mathematical questions. The concept relies on the idea that given a state outside some finite region, the conditional distribution of the Gibbs measure inside the finite region is given by the Boltzmann distribution depending on a local Hamiltonian.

First results investigated existence and uniqueness of Gibbs measures. Dobrushin's condition of weak dependence [Dob68] provides suitable precondition on uniqueness of Gibbs measures in lattice systems. Non-uniqueness, on the other hand, is interpreted as phase transition. Another question concerns conditions which imply that symmetries of the local Hamiltonians are preserved by Gibbs measures. Breaking of symmetry, on the other hand, implies the existence of a phase transition. There are several results in dimension two. The Mermin-Wagner Theorem [MW66] and its more recent variants state preservation of certain continuous symmetries in dimensions one and two such as translations or pure spin transformations. In dimension two other results on preservation of continuous internal symmetries and translation are, among others, due to Dobrushin and Shlosman [DS75] as well as Pfister [Pfis81] in lattice models, and in the case of continuum systems, due to Shlosman [Shl79], Fröhlich and Pfister [FP81], [FP86] and Georgii [Geo99]. The mathematicians D. Ioffe, S. Shlosman and Y. Velenik [ISV02] were able to relax assumptions on the interaction potential and treat the non-smooth case in lattice systems. T. Richthammer generalised it to point processes in [Ric05] and [Ric09]. Contrarily, breaking of rotational symmetry in dimension two at low temperature has been indicated since long, compare with [Mer68] and [NH79]. F. Merkl and S. W. W. Rolles showed in [MR09] the breaking of rotation symmetry in a simple model of two dimensional crystals without defects. In this model of crystals, atoms can be enumerated by a triangular lattice. In the very recent work [HMR13] by M. Heydenreich, F. Merkl and S. W. W. Rolles, defects were integrated into the model. Defects are isolated single missing atoms, however, the results in [HMR13] can be generalised to larger bounded islands of missing atoms as also mentioned in [HMR13], but non-local defects are not included. The first model in [MR09] treated pair potentials with at least quadratic growth; the second one, [HMR13], tackled the case of strictly convex potentials.

In this thesis, we are going to examine an analogue of the models in [MR09] and [HMR13] with a hard-core repulsion. For this potential we show the existence of breaking of rotational symmetry. In Section 3 we address the case without defects; in Section 4 we apply a result from [HMR13] to deal with isolated defects. In Section 5 we carry over the finite-volume result without defects to some infinite-volume Gibbs measure. This work is motivated by the following open problem: is there a Gibbs measure on the set of locally finite point configurations in $\mathbb{R}^2$ which breaks the rotational symmetry of the hard-core potential? This question is analogous to the problem which was solved in [Geo99] and [Ric09] for translational symmetry. However, with another outcome than what is expected in the case of rotational symmetry, since due to [Geo99] and [Ric09], translational symmetry is preserved. We regard our simple semi-discrete model as a very first step in addressing this question. The general problem might be approached from following perspective. Let us remove a finite, connected set of vertices from a triangular lattice with side length $1+\epsilon$ where ϵ is sufficiently small. Take a Poisson point process with sufficiently high activity parameter on an open subset of $\mathbb{R}^2$ which contains this finite connected set and its boundary that is part of the remaining lattice. Conditioned on the event that Poisson points have Euclidean distances greater than one from each other, and so from the boundary, the expected point configuration is close to the scaled triangular lattice which is determined by the missing part of the lattice. This scenario can only happen if the activity parameter is chosen high enough, and for our purpose, this choice of the activity parameter should work for any size of the missing lattice part. With this question in mind, one can recover this idea behind the model and the results in Section 2.

2 The model

2.1 Configuration space

The *standard triangular lattice* in $\mathbb{R}^2$ is the set $I = \mathbb{Z} + \tau\mathbb{Z}$ with $\tau = e^{\frac{i\pi}{3}}$. We identify $\mathbb{Z} \subset \mathbb{R} \subset \mathbb{R}^2$ by $\mathbb{R} \ni x \hat{=} (x, 0) \in \mathbb{R}^2$ and $\mathbb{R}^2 \subset \mathbb{C}$ by $(x, y) \hat{=} x + iy$. We think of I as an index set, which is going to be used to parametrize countable point configurations in the real plane. Let us define the quotient space $I_N = I/(NI)$ for an $N \in \mathbb{N} := \{1, 2, 3, ...\}$. We will think of I_N as the following specific set of representatives:

$$I_N = \{x + y\tau \mid x, y \in \{0, ..., N-1\}\}. \tag{2.1}$$

A *parametrized point configuration* in $\mathbb{R}^2$ is a function $\omega : I \to \mathbb{R}^2$, $x \mapsto \omega(x)$, which determines the point configuration $\{\omega(x) \mid x \in I\} \subset \mathbb{R}^2$. For the set of all parametrized point configurations we introduce the character $\Omega = \{\omega : I \to \mathbb{R}^2\}$. Note that a single point configuration $\{\omega(x) \mid x \in I\} \subset \mathbb{R}^2$ can be parametrized by many different $\omega \in \Omega$.

Let $\epsilon \in (0, 1]$. An N*-periodic parametrized point configuration* with side length $l \in (1, 1+\epsilon)$ is a parametrized configuration ω which satisfies the *periodic boundary conditions*:

$$\omega(x + Ny) = \omega(x) + lNy \quad \text{for all } x, y \in I. \tag{2.2}$$

The set of N-periodic parametrized configurations with side length l is denoted by $\Omega_{N,l}^{per} \subset \Omega$. From now on we will omit the word parametrized because we are going to work solely with *point configurations* which are parametrized by I. An N-periodic configuration is uniquely determined by its values on I_N. Therefore, we identify N-periodic configurations $\omega \in \Omega_{N,l}^{per}$ with functions $\omega : I_N \to \mathbb{R}^2$.

The bond set $E \subset I \times I$ contains index-pairs with Euclidean distance one; this is $E = \{(x, y) \in I \times I \mid |x - y| = 1\}$. In order to transfer the definition to the quotient space I_N, we define an equivalence relation $\sim_N$ on E by $(x, y) \sim_N (x', y')$ if and only if there is a $z \in NI$ such that $x = x' + z$ and $y = y' + z$. Now, set $E_N = E/\sim_N$. We can think of E_N as a bond set $E_N \subset I_N \times I_N$.

For $x \in I$ and $z \in \{1, \tau\}$, define the open *triangle*

$$\triangle_{x,z} = \{x + sz + t\tau z \mid 0 < s, t,\ s + t < 1\}$$

with corner points x, $x + z$ and $x + \tau z$. For $\triangle_{x,z}$ denote the set of corner points by $\mathcal{S}(\triangle_{x,z}) = \{x, x + z, x + \tau z\}$. On the set of all triangles

$$\mathcal{T} = \{\triangle_{x,z} \mid x \in I \text{ and } z \in \{1, \tau\}\},$$

we define an equivalence relation: $\triangle_{x,z} \sim_N \triangle_{x',z'}$ if and only if $x - x' \in NI$ and $z = z'$. The set of equivalence classes is denoted by $\mathcal{T}_N = \mathcal{T}/\sim_N$. We identify equivalence classes

$\triangle \in \mathcal{T}_N$ with their unique representative with corners in the set $\{x+\tau y \mid x, y \in \{0, ..., N\}\}$. The closures of the triangles in $\mathcal{T}_N$ cover the convex hull of the above set, which we denote by $U_N = \text{conv}(\{x + \tau y \mid x, y \in \{0, ..., N\}\})$.

2.2 Probability space

Due to the definitions of Ω and $\Omega^{per}_{N,l}$, we can identify $\Omega = (\mathbb{R}^2)^I$ and $\Omega^{per}_{N,l} = (\mathbb{R}^2)^{I_N}$. Both sets are endowed with the corresponding product σ-fields $\mathcal{F} = \bigotimes_{x\in I} \mathcal{B}(\mathbb{R}^2)$ and $\mathcal{F}_N = \bigotimes_{x\in I_N} \mathcal{B}(\mathbb{R}^2)$ where $\mathcal{B}(\mathbb{R}^2)$ denotes the Borel σ-field on each factor. The event of admissible, N-periodic configurations $\Omega_{N,l} \subset \Omega^{per}_{N,l}$ is defined by the properties $(\Omega 1) - (\Omega 3)$:

(Ω1) $|\omega(x) - \omega(y)| \in (1, 1 + \epsilon)$ for all $(x, y) \in E$.

For $\omega \in \Omega$ we define the extension $\hat{\omega} : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\hat{\omega}(x) = \omega(x)$ if $x \in I$, and on the closure of any triangle $\triangle \in \mathcal{T}$, the map $\hat{\omega}$ is defined to be the unique affine linear extension of the mapping defined on the corners of $\triangle$.

(Ω2) The map $\hat{\omega} : \mathbb{R}^2 \to \mathbb{R}^2$ is injective.

(Ω3) The map $\hat{\omega}$ is orientation preserving, this is to say that $\det(\nabla\hat{\omega}(x)) > 0$ for all $\triangle \in \mathcal{T}$ and $x \in \triangle$ with the Jacobian $\nabla\hat{\omega} : \cup\mathcal{T} \to \mathbb{R}^{2\times 2}$.

Define the set of *admissible, N-periodic configurations* as

$$\Omega_{N,l} = \{\omega \in \Omega^{per}_{N,l} \mid \omega \text{ satisfies } (\Omega 1)\text{–}(\Omega 3)\}$$

and the set of all *admissible configurations* as $\Omega_\infty = \{\omega \in \Omega \mid \omega \text{ satisfies } (\Omega 1)\text{–}(\Omega 3)\}$. Note that for $\omega \in \Omega^{per}_{N,l}$, ($\Omega$2) is fulfilled if and only if $\hat{\omega}$ is a bijection; this is a consequence of the periodic boundary conditions (2.2).

The set $\Omega_{N,l}$ is non-empty and open in $(\mathbb{R}^2)^{I_N}$. The scaled standard configuration $\omega_l(x) = lx$, for $x \in I$ and $1 < l < 1 + \epsilon$, is an element both of $\Omega_{N,l}$ and Ω_∞. Figure 1 illustrates a part of some admissible, 4-periodic configuration. The points of the configuration are illustrated by hard disks with radii 1/2. The image of I_4 and those of two equivalent triangles are shaded.

Clearly, $0 < \delta_0 \otimes \lambda^{I_N\setminus\{0\}}(\Omega_{N,l}) < \infty$ with the Lebesgue measure λ on $\mathbb{R}^2$ and the Dirac measure δ_0 in $0 \in \mathbb{R}^2$. The lower bound holds because sections of $\Omega_{N,l}$ are non-empty and open in $(\mathbb{R}^2)^{I_N\setminus\{0\}}$ if $\omega(0)$ is fixed; the upper bound is a consequence of the parameter ϵ in (Ω1). Let the probability measure $P_{N,l}$ be

$$P_{N,l}(A) = \frac{\delta_0 \otimes \lambda^{I_N\setminus\{0\}}(\Omega_{N,l} \cap A)}{\delta_0 \otimes \lambda^{I_N\setminus\{0\}}(\Omega_{N,l})}$$

for any Borel measurable set $A \in \mathcal{F}_N$, thus $P_{N,l}$ is the uniform distribution on the set $\Omega_{N,l}$ with respect to the *reference measure* $\delta_0 \otimes \lambda^{I_N\setminus\{0\}}$. The first factor in this product refers to the component $\omega(0)$ of $\omega \in \Omega$.

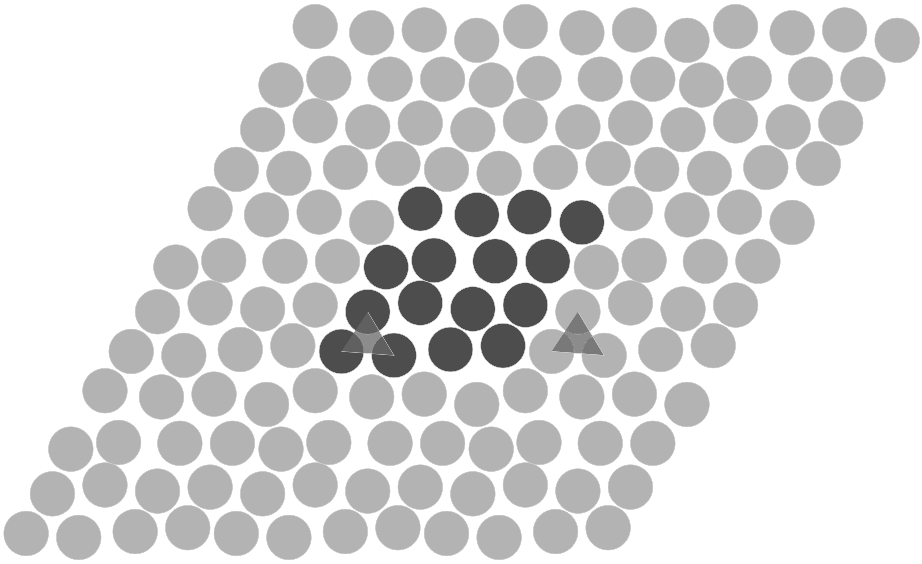

Figure 1: A part of an admissible 4-periodic configuration.

2.3 Results

We have the following finite-volume result.

Theorem 2.1. *For ϵ sufficiently small (such that equation (3.7) holds for all $1 < a_i < 1+\epsilon$), one has*

$$\lim_{l\downarrow 1} \sup_{N\in\mathbb{N}} \sup_{\triangle\in\mathcal{T}_N} E_{P_{N,l}}[\ |\nabla\hat{\omega}(\triangle) - \mathrm{Id}|^2\ ] = 0 \tag{2.3}$$

with the constant value of the Jacobian $\nabla\hat{\omega}(\triangle)$ on the set $\triangle \in \mathcal{T}_N$ and any norm $|\cdot|$ on $\mathbb{R}^{2\times 2}$.

Since the convergence in *Theorem 2.1* is uniform in N, we can find an infinite-volume Gibbs measure P such that $E_P[\ |\nabla\hat{\omega}(\triangle) - \mathrm{Id}|^2\ ]$ is small on every triangle $\triangle \in \mathcal{T}$. This observation is object of the following theorem.

Theorem 2.2. *In the case of ϵ as given in* Theorem 2.1*, for all $\delta > 0$ there is an infinite-volume Gibbs measure P, associated with the hard-core Hamiltonian on the set Ω_∞, which has the property $P(\Omega_\infty) = 1$ and fulfils*

$$\sup_{\triangle\in\mathcal{T}} E_P[\ |\nabla\hat{\omega}(\triangle) - \mathrm{Id}|^2\ ] \leq \delta. \tag{2.4}$$

In fact, it is a result about a spontaneous breaking of the rotational symmetry in a strong sense. The set Ω_∞ is rotational invariant, and this symmetry is broken by the Gibbs measure P in the sense of (2.4). The exact definition of a Gibbs measure will be given in Section 5. The central argument is going to be the following rigidity theorem from [FJM02, Theorem 3.1], which generalises Liouville's Theorem.

Theorem 2.3 (Friesecke, James and Müller)**.** *Let U be a bounded Lipschitz domain in $\mathbb{R}^n$, $n \geq 2$. There exists a constant $C(U)$ with the following property: For each $v \in W^{1,2}(U, \mathbb{R}^n)$ there is an associated rotation $R \in \mathrm{SO}(n)$ such that*

$$||\nabla v - R||_{L^2(U)} \leq C(U)||\mathrm{dist}(\nabla v, \mathrm{SO}(n))||_{L^2(U)}.$$

Liouville's Theorem states that a function v, fulfilling $\nabla v(x) \in \mathrm{SO}(n)$ almost everywhere, is a rigid motion. Indeed, *Theorem 2.3* generalises this result. We are going to set $v = \hat{\omega}|_{U_N}$ and $U = U_N$, which is a bounded Lipschitz domain. The function $\hat{\omega}|_{U_N}$ is affine linear on each triangle $\triangle \in \mathcal{T}_N$, thus piecewise affine linear on U_N. As a consequence, $\hat{\omega}|_{U_N}$ belongs to the class $W^{1,2}(U_N, \mathbb{R}^n)$. The following remark, which also appears in [HMR13, Remark 1.5], is essential to achieve uniformity in *Theorem 2.1* in the parameter N.

Remark 2.4. *The constant $C(U)$ in* Theorem 2.3 *is invariant under scaling of the domain: $C(\alpha U) = C(U)$ for all $\alpha > 0$. In fact, setting $v_\alpha(\alpha x) = \alpha v(x)$ for $x \in U$, we have $\nabla v_\alpha(\alpha x) = \nabla v(x)$, and therefore $||\nabla v_\alpha - R||_{L^2(\alpha U)} = \alpha^{n/2}||\nabla v - R||_{L^2(U)}$, and $||\mathrm{dist}(\nabla v_\alpha, \mathrm{SO}(n))||_{L^2(\alpha U)} = \alpha^{n/2}\, ||\mathrm{dist}(\nabla v, \mathrm{SO}(n))||_{L^2(U)}$. Consequently, the constants $C(U_N)$ for the domains U_N $(N \geq 1)$ can be chosen independently of N.*

Spontaneous breaking of the rotational symmetry in the usual sense can be proved easier. This observation is formulated and proved in the next proposition. A similar result and its proof is also mentioned in [HMR13, Section 1.3].

Proposition 2.5. *For all $l \in (1, 1+\epsilon)$, $N \in \mathbb{N}$, $x \in I$ and $z \in I$ with $(0, z) \in E$, we have*

$$E_{P_{N,l}}[\omega(x+z) - \omega(x)] = lz. \tag{2.5}$$

Proof. We follow the ideas remarked in [HMR13, Section 1.3]. The reference measure $\delta_0 \otimes \lambda^{I_N \setminus \{0\}}$ is invariant under the bijective translations

$$\psi_b : \Omega_{N,l}^{per} \to \Omega_{N,l}^{per} \quad (\omega(x))_{x \in I} \mapsto (\omega(x+b) - \omega(b))_{x \in I} \tag{2.6}$$

for all $b \in I$. The set $\Omega_{N,l}$ is also invariant under $\psi_b^{-1} = \psi_{-b}$. As a consequence, the measures $P_{N,l}$ are invariant under ψ_b for all $b \in I$, and the random vectors $\omega(x+z) - \omega(x)$ have the same distribution under $P_{N,l}$ for all $x \in I$ and a fixed z. Therefore, we obtain (2.5) from the periodic boundary conditions (2.2). □

The expression $|\omega(x+z) - \omega(x)|$ is $P_{N,l}$-almost surely uniformly bounded in N, hence (2.5) carries over to weak limits of $P_{N,l}$ as $N \to \infty$. Consequently, such weak limits are not rotational invariant. We will show in Section 5 that there is a Gibbs measure which, on finite subsets $\Lambda \subset I$, is a weak limit of some subsequence of $(P_{N,l})_{N \in \mathbb{N}}$. By the above remark, this Gibbs measure breaks the rotational invariance of the set Ω_∞. However, in Section 3 we show *Theorem 2.1*, which states symmetry breaking in a much stronger sense. Subsequently, in Section 5 we show *Theorem 2.2*, the existence of an infinite-volume Gibbs measure which inherits this property.

3 Finite-volume arguments

We are going to show that for $\omega \in \Omega_{N,l}$, the L^2-distance on U_N of the Jacobian matrix $\nabla\hat{\omega}$ from the scaled identity matrix l Id can be controlled by the difference of the areas of $\hat{\omega}(U_N)$ and U_N. Because of the periodic boundary conditions, $\lambda(\hat{\omega}(U_N))$ does not depend on configurations ω with (Ω2), thus the mentioned area difference provides a suitable uniform control on the set $\Omega_{N,l}$. First, we show that the L^2-distance of $\nabla\hat{\omega}$ from the scaled identity l Id can be controlled by the sum over the squared deviations of the triangles' side lengths from one. The one should be associated with the side length of an equilateral triangle.

The following lemma provides the desired estimate on a single triangle. It states that the distance from SO(2) of a linear map near SO(2) can be controlled by terms which measure how the linear map deforms the side lengths of a standard, equilateral triangle. We take ideas for the proof from [Th06, Lemma 4.2. in the appendix].

Lemma 3.1. *There is a positive constant C such that, for all linear maps $A : \mathbb{R}^2 \to \mathbb{R}^2$ with $\det(A) > 0$ and the property*

$$||Av_i| - 1| \leq 1 \quad \textit{for all } i \in \{1, 2, 3\}$$

where $v_1 = (1, 0)$, $v_2 = (\frac{1}{2}, \frac{\sqrt{3}}{2})$, $v_3 = v_1 - v_2$, the following inequality holds:

$$\operatorname{dist}(A\ ,\ \mathrm{SO}(2))^2 := \inf_{R\in \mathrm{SO}(2)} |A - R|^2 \leq C \max_{i\in\{1,2,3\}} ||Av_i| - 1|^2 \tag{3.1}$$

where $|M| = \sqrt{\operatorname{tr}(M^tM)}$ is the Frobenius norm and $|v|$ is the Euclidean norm of v. Conversely, the inequality

$$\max_{i\in\{1,2,3\}} ||Av_i| - 1|^2 \leq \operatorname{dist}(A\ ,\ \mathrm{SO}(2))^2 \tag{3.2}$$

holds for every linear $A : \mathbb{R}^2 \to \mathbb{R}^2$.

Proof. To show the second inequality, we compute with the reverse triangle inequality:

$$\begin{aligned} \max_{i\in\{1,2,3\}} ||Av_i| - 1| &= \inf_{R\in \mathrm{SO}(2)} \max_{i\in\{1,2,3\}} ||Av_i| - |Rv_i|| \leq \inf_{R\in \mathrm{SO}(2)} \max_{i\in\{1,2,3\}} |(A - R)v_i| \\ &\leq \inf_{R\in \mathrm{SO}(2)} \max_{i\in\{1,2,3\}} |A - R|\ |v_i| = \operatorname{dist}(A, \mathrm{SO}(2)). \end{aligned}$$

To prove (3.1), first we are going to show that $\inf_{R\in\mathrm{SO}(2)} |A - R|$ is given by $|\sqrt{A^tA} - \mathrm{Id}|$. Observe that for $U \in \mathrm{SO}(2)$, the Frobenius norm $|M|$ of the matrix M is invariant under right and left multiplication by U, meaning $|M| = |UM| = |MU|$. By the Singular Value Decomposition Theorem, there are $U, V \in \mathrm{SO}(2)$ such that $D = V^t\sqrt{A^tA}\ V = U^tAV$ is a diagonal, positive-definite matrix. Here we made use of the assumption $\det A > 0$. Hence, by the above invariance property, we can rearrange: $\inf_{R\in \mathrm{SO}(2)} |A-R| = \inf_{R\in \mathrm{SO}(2)} |D-R|$ and compute

$$\begin{aligned} |D-R|^2 = \mathrm{tr}((D-R)^t(D-R)) &= \mathrm{tr}(D^2) + \mathrm{tr}(\mathrm{Id}) - 2\ \mathrm{tr}(DR) \\ &\geq \mathrm{tr}(D^2) + \mathrm{tr}(\mathrm{Id}) - 2\ \mathrm{tr}(D) \\ &= |D - \mathrm{Id}|^2. \end{aligned}$$

This inequality proves the equation $\inf_{R\in SO(2)} |A-R| = |D-\mathrm{Id}| = |\sqrt{A^tA} - \mathrm{Id}|$. Like in [Th06], we define

$$m := \max_{i=1,2,3} ||Av_i|^2 - 1| \quad \text{and} \quad p := |\sqrt{A^tA} - \mathrm{Id}|.$$

We proceed similarly to the proof of [Th06, Lemma 4.2 (Appendix)]. It suffices to prove that $p \leq Km$ for some positive constant K, because $|x^2-1| \leq 3|x-1|$ for $0 \leq x \leq 2$, and thus

$$p^2 \leq K^2m^2 \leq 9K^2 \max_{i=1,2,3} ||Av_i| - 1|^2,$$

which is the desired inequality with $C = 9K^2$. Let us also define the matrix $G = A^tA - \mathrm{Id}$. With G we can write $||Av_i|^2 - 1| = |\langle v_i, Gv_i\rangle| \leq m$ for all $i \in \{1,2,3\}$ by the definition of m. With the (reverse) triangle inequality, we compute: $m \geq |\langle v_3, Gv_3\rangle| \geq 2\ |\langle v_2, Gv_1\rangle| - |\langle v_1, Gv_1\rangle| - |\langle v_2, Gv_2\rangle|$, thus $\frac{3}{2}m \geq |\langle v_i, Gv_j\rangle|$ for all $i,j \in \{1,2,3\}$. By this result, we conclude that there is a positive constant $K > 0$ such that $|G| \leq (K/\sqrt{2})\ m$. By the equality $x^2-1 = (x+1)(x-1)$, we can estimate $p = |G(\sqrt{A^tA}+\mathrm{Id})^{-1}| \leq |G|\ |(\sqrt{A^tA}+\mathrm{Id})^{-1}| < Km$ because $|(\sqrt{A^tA}+\mathrm{Id})^{-1}| = |(D+\mathrm{Id})^{-1}| < \sqrt{2}$ by the positive-definiteness of D. Note that, since $0 < \det(D+\mathrm{Id}) = \det(\sqrt{A^tA}+\mathrm{Id})$, $\sqrt{A^tA}+\mathrm{Id}$ is invertible. □

Now, we prove an estimate, which provides control over the L^2-distance of $\nabla\hat{\omega}$ from the scaled identity matrix in terms of the side length deviations.

Lemma 3.2. *There is a constant c_2 such that for all $N \geq 1$ and $1 < l < 1+\epsilon$ the inequality*

$$||\ \nabla\hat{\omega} - l\ \mathrm{Id}\ ||^2_{L^2(U_N)} \leq c_2 \sum_{(x,y)\in E_N} (|\omega(x) - \omega(y)| - 1)^2 \tag{3.3}$$

holds for all $\omega \in \Omega_{N,l}$, and hence

$$E_{P_{N,l}}[\ ||\ \nabla\hat{\omega} - l\ \mathrm{Id}\ ||^2_{L^2(U_N)}\ ] \leq c_2 \sum_{(x,y)\in E_N} E_{P_{N,l}}[\ (|\omega(x) - \omega(y)| - 1)^2\ ] \tag{3.4}$$

where the L^2-norm is defined with respect to some scalar product on $\mathbb{R}^{2\times 2}$, and $|\cdot|$ denotes the Euclidean norm on $\mathbb{R}^2$.

Note that the right side in equation (3.3) is strictly positive because of the boundary conditions (2.2) and because $l > 1$, whereas the left is zero for $\omega = \omega_l \in \Omega^{per}_{N,l}$. Since the measure $P_{N,l}$ is supported on the set $\Omega_{N,l}$, (3.4) follows from (3.3). Also note that c_2 does not depend on N.

Proof. Let $\omega \in \Omega_{N,l}$. By *Lemma 3.1* we conclude that on every triangle $\triangle \in \mathcal{T}_N$, we have

$$\operatorname{dist}\left(\nabla\hat{\omega}(\triangle),\ \mathrm{SO}(2)\right)^2 \leq C \max_{x \neq y \in S(\triangle)} (|\omega(x) - \omega(y)| - 1)^2 \leq \frac{C}{2} \sum_{x \neq y \in S(\triangle)} (|\omega(x) - \omega(y)| - 1)^2$$

where we used the assumption $\epsilon \leq 1$ together with (Ω1) and (Ω3) to apply *Lemma 3.1*. The factor 1/2 is a consequence of summing over all non-equal pairs (x, y). Orthogonality of the functions which are non-zero on different triangles gives

$$||\ \operatorname{dist}(\nabla\hat{\omega}, \mathrm{SO}(2))\ ||^2_{L^2(U_N)} \leq c_{21} \sum_{(x,y) \in E_N} (|\omega(x) - \omega(y)| - 1)^2$$

with $c_{21} = C\ \lambda(\triangle_{0,1}) = C\sqrt{3}/4$ because we sum again over both pairs (x, y) and (y, x) on the right side. With application of *Theorem 2.3* about geometric rigidity, we find an $R(\omega) \in \mathrm{SO}(2)$ such that

$$||\ \nabla\hat{\omega} - R(\omega)\ ||^2_{L^2(U_N)} \leq c_{22}\ ||\ \operatorname{dist}(\nabla\hat{\omega}, \mathrm{SO}(2))\ ||^2_{L^2(U_N)},$$

with a constant c_{22}, which does not depend on N by Remark 2.4. Because of the periodic boundary conditions (2.2), the function $\hat{\omega} - l\ \mathrm{Id}$ is N-periodic, this is to say

$$\hat{\omega}(x + Ny) - l(x + Ny) = \hat{\omega}(x) - lx \quad \text{for all } x \in \mathbb{R}^2 \text{ and } \ y \in I. \tag{3.5}$$

Let $A \in \mathbb{R}^{2\times 2}$ be a constant matrix. Integrating the function $\langle \nabla\hat{\omega} - l\ \mathrm{Id}, A\rangle$ over the set U_N, the result equals zero since, by (3.5) and the Fundamental Theorem of Calculus,

$$\int_0^1 \langle \nabla\hat{\omega} - l\ \mathrm{Id}, A\rangle(x + tN)\mathrm{d}t = 0 \quad \text{for all } x \in \mathbb{R}^2$$

where we used the embedding $\mathbb{R} \subset \mathbb{R}^2$. Consequently, we obtain the orthogonality property: $\nabla\hat{\omega} - l\ \mathrm{Id} \perp_{L^2(U_N)} A$, for any constant matrix $A \in \mathbb{R}^{2\times 2}$ and thus

$$||\ \nabla\hat{\omega} - l\ \mathrm{Id}\ ||^2_{L^2(U_N)} + ||\ l\ \mathrm{Id} - R(\omega)\ ||^2_{L^2(U_N)} = ||\ \nabla\hat{\omega} - R(\omega)\ ||^2_{L^2(U_N)}$$

by Pythagoras. Since $||\ l\ \mathrm{Id} - R(\omega)\ ||^2_{L^2(U_N)} \geq 0$ and because $P_{N,l}$ is supported on the set $\Omega_{N,l}$, the lemma is established with $c_2 = c_{21}c_{22}$. □

With *Lemma 3.2* we can now prove *Theorem 2.1.*

Proof of Theorem 2.1. Heron's formula states that the area $\lambda(\triangle)$ of the triangle $\triangle$ with side lengths a_1, a_2, a_3 is given by

$$\lambda(\triangle) = \frac{1}{4}\sqrt{(a_1+a_2+a_3)(-a_1+a_2+a_3)(a_1-a_2+a_3)(a_1+a_2-a_3)}. \tag{3.6}$$

We obtain by first order Taylor approximation of (3.6) at the point $a_i = 1$, $i \in \{1,2,3\}$ that

$$\lambda(\triangle) - \lambda(\triangle_{0,1}) = \frac{1}{2\sqrt{3}}\sum_{i=1}^{3}(a_i - 1) + o\left(\sum_{i=1}^{3}|a_i - 1|\right) \quad \text{as } (a_1,a_2,a_3) \to (1,1,1).$$

Since the function λ is smooth in a neighbourhood of $(1,1,1)$, we could also express the remainder term as Big $\mathcal{O}$ of the sum of the squares. In the following we only need the weaker estimate on the remainder. We choose ϵ so small that the inequality

$$\frac{1}{4\sqrt{3}}\sum_{i=1}^{3}(a_i - 1) \leq \lambda(\triangle) - \lambda(\triangle_{0,1}) \tag{3.7}$$

is satisfied whenever $1 < a_i < 1 + \epsilon$. Note that we have divided the constant by two preceding the sum. Let us fix such an ϵ and assume that $\Omega_{N,l}^{per}$ is defined by means of this ϵ. Using (3.7), we can also estimate the squared side length deviations:

$$\sum_{i=1}^{3}(a_i - 1)^2 \leq 4\sqrt{3}\,\epsilon\,(\lambda(\triangle) - \lambda(\triangle_{0,1})). \tag{3.8}$$

By equation (3.3) from *Lemma 3.2* and (3.8), we get an upper bound for $||\nabla\hat{\omega} - l\ \mathrm{Id}||_{L^2(U_N)}^2$ in terms of the area differences. By summing up the contributions (3.8) of the triangles $\triangle \in \mathcal{T}_N$, we conclude for all $\omega \in \Omega_{N,l}$ that

$$||\ \nabla\hat{\omega} - l\ \mathrm{Id}\ ||_{L^2(U_N)}^2 \leq 4\sqrt{3}\,\epsilon\,c_2 \sum_{\triangle\in\mathcal{T}_N}(\lambda(\hat{\omega}(\triangle)) - \lambda(\triangle_{0,1})). \tag{3.9}$$

Because of (Ω2) and the periodic boundary conditions (2.2), the right hand side in (3.9) does not depend on $\omega \in \Omega_{N,l}$. Hence, with $\omega_l \in \Omega_{N,l}$ we can compute

$$\sum_{\triangle\in\mathcal{T}_N}(\lambda(\hat{\omega}(\triangle)) - \lambda(\triangle_{0,1})) = \sum_{\triangle\in\mathcal{T}_N}(\lambda(\hat{\omega}_l(\triangle)) - \lambda(\triangle_{0,1})) = |\mathcal{T}_N|\ \lambda(\triangle_{0,1})(l^2 - 1). \tag{3.10}$$

The combination of the equations (3.9) and (3.10) gives

$$|| \nabla\hat{\omega} - l \text{ Id } ||^2_{L^2(U_N)} \leq 4\sqrt{3}\,\epsilon\, c_2\ |\mathcal{T}_N|\ \lambda(\triangle_{0,1})(l^2-1). \tag{3.11}$$

The reference measure $\delta_0 \otimes \lambda^{I_N\setminus\{0\}}$ and the set of allowed configurations $\Omega_{N,l}$ are invariant under the reflection $\phi : \omega \mapsto (-\omega(-x))_{x\in I}$ and the translations ψ_b for $b \in I$, defined in (2.6). As a consequence, the measure $P_{N,l}$ is also invariant under these maps, and therefore the matrix valued random variables $\nabla(\hat{\omega}(\triangle))$ are identically distributed for all $\triangle \in \mathcal{T}_N$. Thus, for all $\triangle \in \mathcal{T}_N$, one has

$$E_{P_{N,l}}[\ || \nabla\hat{\omega} - l \text{ Id } ||^2_{L^2(U_N)}\ ] = |\mathcal{T}_N|\ \lambda(\triangle_{0,1}) E_{P_{N,l}}[\ |\nabla\hat{\omega}(\triangle) - l \text{ Id}|^2\ ].$$

This equation, together with (3.11), implies

$$\lim_{l\downarrow 1} \sup_{N\in\mathbb{N}} \sup_{\triangle\in\mathcal{T}_N} E_{P_{N,l}}[\ |\nabla\hat{\omega}(\triangle) - l \text{ Id}|^2\ ] = 0.$$

By means of the triangle inequality, we see that for all $\triangle \in \mathcal{T}_N$ and $\omega \in \Omega_{N,l}$

$$|\nabla\hat{\omega}(\triangle) - \text{Id}|^2 \leq |\nabla\hat{\omega}(\triangle) - l \text{ Id}|^2 + c_3^2(l-1)^2 + 2c_3\ |l-1|\ |\nabla\hat{\omega}(\triangle) - l \text{ Id}|$$

with $c_3 = |\text{Id}| > 0$. For $\omega \in \Omega_{N,l}$, the term $|\nabla\hat{\omega}(\triangle) - l \text{ Id}|$ is uniformly bounded for $l \in (1, \epsilon)$ and $N \in \mathbb{N}$, which proves the theorem. □

4 Crystals with defects

We can generalise *Theorem 2.1* for crystals with local defects like in [HMR13]. A defect is a missing point in the configuration $\omega \in \Omega_\infty$. We are going to work in a similar model as in [HMR13] and require that every nearest and next-nearest neighbour of defects is present. This requirement implies that there is a solid ring around every defect which is formed by triangles. This ring is solid in the following sense: if the triangles in the ring have small side lengths greater than one, then the hexagon around the missing value is close to some regular hexagon with side length one. By this approach we can assure that by merely requiring present edges to be small, also missing edges can be made small, and hence we end up with the same configurations as in *Theorem 2.1*.

In this section we modify the definition of (parametrized) point configurations as follows. A *point configuration* in $\mathbb{R}^2$ is a function $\omega : I \to \mathbb{R}^2 \cup \{\hexagon\}$, $x \mapsto \omega(x)$, which determines the point configuration $\{\omega(x) \mid x \in I \text{ and } \omega(x) \neq \hexagon\} \subset \mathbb{R}^2$. The hexagon symbol, $\hexagon \notin \mathbb{R}^2$, represents an empty space, this is to say that there is a *hole*, *defect* or *missing point* in the lattice at $x \in I$ whenever $\omega(x) = \hexagon$. The set of all parametrized point configurations is again denoted by the character $\Omega = \{\omega : I \to \mathbb{R}^2 \cup \{\hexagon\}\}$. The set of *N-periodic configurations* $\Omega^{per}_{N,l}$ consists of parametrized point configurations which satisfy

$$\hat{\omega}(x + Ny) = \hat{\omega}(x) + lNy \quad \text{for all } x, y \in I \tag{4.1}$$

where the function $\hat{\omega} : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$\hat{\omega}(x) = \begin{cases} \omega(x) & \text{if } x \in I \text{ and } \omega(x) \neq \hexagon \\ \\ \frac{1}{6}\sum_{k=1}^{6} \omega(x + e^{\frac{i\pi k}{3}}) & \text{if } x \in I \text{ and } \omega(x) = \hexagon \end{cases}$$

for $x \in I$. On the closure of any triangle $\triangle \in \mathcal{T}$, the map $\hat{\omega}$ is defined to be the unique affine linear extension of the above mapping. Note that $\hat{\omega}$ is well defined if and only if nearest neighbours of missing points are present. The event of admissible, N-periodic configurations $\Omega_{N,l} \subset \Omega^{per}_{N,l}$ is defined analogously to the case without defects:

($\widetilde{\Omega 0}$) Defects are isolated in the sense that $\omega(x) = \hexagon$ and $\omega(y) = \hexagon$ only if $|x - y| > 2$.

($\widetilde{\Omega 1}$) $|\omega(x) - \omega(y)| \in (1, 1 + \epsilon)$ for all $(x, y) \in E$ with $\omega(x) \neq \hexagon$ and $\omega(y) \neq \hexagon$.

($\widetilde{\Omega 2}$) The map $\hat{\omega} : \mathbb{R}^2 \to \mathbb{R}^2$ is injective.

($\widetilde{\Omega 3}$) The map $\hat{\omega}$ is orientation preserving, this is to say that $\det(\nabla\hat{\omega}(x)) > 0$ for all $\triangle \in \mathcal{T}$ and $x \in \triangle$.

The set of *admissible, N-periodic configurations* $\Omega_{N,l}$ is then defined as

$$\Omega_{N,l} = \{\omega \in \Omega^{per}_{N,l} \mid \omega \text{ satisfies } (\widetilde{\Omega 0})\text{–}(\widetilde{\Omega 3})\}.$$

An important difference from the case without defects is that $(\widetilde{\Omega 1})$ does not require missing points to have a described distance to other points, for instance in terms of $\hat{\omega}$, which determines a location for missing points. Also define the sets

$$\begin{aligned}
\Omega_\infty &= \{\omega \in \Omega \mid \omega \text{ satisfies } (\widetilde{\Omega 0})\text{–}(\widetilde{\Omega 3})\}, \\
\Omega_\infty^0 &= \{\omega \in \Omega_\infty \mid \omega(0) \in \{0, \varnothing\}, \text{ and } \omega(\tau) = 0 \text{ whenever } \omega(0) = \varnothing\} \text{ and} \\
\Omega_{N,l}^0 &= \Omega_{N,l} \cap \Omega_\infty^0.
\end{aligned}$$

Now, we construct probability measures $P_{N,m,l}$ on $\Omega_{N,l}$. Recall the translations ψ_b from (2.6). In order to prove the defect-case analogue of *Theorem 2.1*, we have to construct $P_{N,m,l}$ which are invariant under translations of the (2.6) type. Such an invariance is needed because we wish that the Jacobians $\nabla\hat{\omega}(\triangle)$ of the triangles are identically distributed under $P_{N,m,l}$, which was also used in the proof of *Theorem 2.1* for $P_{N,l}$. In the case of defects, a first guess of an analogue of $\delta_0 \otimes \lambda^{I_N \setminus \{0\}}$ would be the reference measure $(\delta_0 + \delta_{\varnothing}) \otimes (\lambda + \delta_{\varnothing})^{I_N \setminus \{0\}}$. However, the restriction of this measure to $\Omega_{N,l}$ is not invariant under the translations

$$\psi: \ \Omega_{N,l} \to \Omega_{N,l}, \quad (\omega(x))_{x \in I_N} \mapsto \begin{cases} (\omega(x) - \omega(0))_{x \in I_N} & \text{if } \omega(0) \neq \varnothing \\ (\omega(x) - \omega(\tau))_{x \in I_N} & \text{if } \omega(0) = \varnothing \end{cases} \tag{4.2}$$

where $\varnothing - z = \varnothing$ for any $z \in \mathbb{R}^2$. In fact, this is a consequence of the definition $\varnothing - z = \varnothing$, which implies that defects can gain infinite weight after the translation. Note that because of $(\widetilde{\Omega 0})$ the map ψ is well defined.

We follow the construction in [HMR13] and obtain $P_{N,m,l}$ which are shift invariant. Two configurations ω, $\omega' \in \Omega_{N,l}^{per}$ are identified if there is a $z \in I$ such that for all $x \in I$ we have $\omega(x) = \omega'(x + Nz)$. Let $\underline{\Omega}_{N,l}$ be the quotient space with respect to the equivalence relation given by this identification. We can identify the space $\underline{\Omega}_{N,l}$ with a measurable set of representatives $\underline{\Omega}_{N,l} \subset \Omega_{N,l}^{per}$. According to [HMR13], a possible choice of a representative is given by $\omega \in \Omega_{N,l}^{per}$ for which $\omega(z) \in [0, lN) + \tau[0, lN) = \Lambda_{lN}$ for the lexicographically smallest $z \in \{x + \tau y \mid x, y \in \{0, ..., N-1\}\}$ with $\omega(z) \neq \varnothing$.

We endow $\Omega_{N,l}^{per}$ with the reference measure $(\lambda + \delta_{\varnothing})^{I_N}$. The restriction of this measure to a measurable set of representatives $\underline{\Omega}_{N,l}$ defines a *reference measure* μ_N on $\underline{\Omega}_{N,l}$. By construction μ_N is shift invariant, this is to say $\theta_b[\mu_N] = \mu_N$ with the shift operator $\theta_b((\omega(x))_{x \in I}) = (\omega(x - b))_{x \in I}$ for any $b \in I$.

Let us define the *Hamiltonian* in analogy to the equation [HMR13, (1.11)]. For $m \in \mathbb{R} \cup \{\infty\}$ and $\omega \in \Omega_{N,l}^{per}$, we define

$$H_{N,m}(\omega) = \infty \cdot \mathbb{1}_{(\Omega_{N,l})^c}(\omega) + m \sum_{x \in I_N} \mathbb{1}_{\{\omega(x) = \varnothing\}} \tag{4.3}$$

with the complement $(\Omega_{N,l})^c$ of the set $\Omega_{N,l}$. The parameter m plays the role of a chemical potential. It gives information about the amount of energy that is needed to remove a particle. If we integrate $\exp(-H_{N,m})$ with respect to the reference measure μ_N, we obtain the *partition sum*

$$Z_{N,m,l} = \int_{\Omega^{per}_{N,l}} e^{-H_{N,m}(\omega)} \mu_N(\mathrm{d}\omega). \tag{4.4}$$

By $(\widetilde{\Omega 1})$ and $\emptyset \neq \Omega_{N,l} \cap \{\omega(x) \neq \bigcirc\ \forall x \in I_N\}$, the partition sum satisfies $0 < Z_{N,m,l} < \infty$. We use the convention $\exp(-\infty) = 0$. The probability measures $P_{N,m,l}$ are defined to be

$$P_{N,m,l}(\mathrm{d}\omega) = \frac{1}{Z_{N,m,l}} e^{-H_{N,m}(\omega)} \mu_N(\mathrm{d}\omega). \tag{4.5}$$

In the case of $m = \infty$, we have $\psi[P_{N,m=\infty,l}] = P_{N,l}$ with the measures $P_{N,l}$ from Section 2 where ψ denotes the translation (4.2). Note that, in the case $m = \infty$, the probability under $P_{N,m=\infty,l}$ for a defect to occur is zero. Let us define the translated measures $P^0_{N,m,l} = \psi[P_{N,m,l}]$ in general. The remark above yields $P^0_{N,m=\infty,l} = P_{N,l}$.

Again *Theorem 2.3* plays the central role in the proof. The following lemma from [HMR13] states that the sum of squared distances $\mathrm{dist}(\nabla\hat{\omega}(\triangle), \mathrm{SO}(2))^2$ over present triangles around a hole bound the analogue sum over the six missing triangles. To be more precise, we quote [HMR13, Definition 2.5].

Definition. *For $x \in I$, let $U_0(x) := \{\triangle \in \mathcal{T} \mid x \in \mathrm{closure}(\triangle)\}$ denote the set of all triangles in $\mathcal{T}$ adjacent to x and let $\mathcal{N} = \{\tau^j \mid j \in \mathbb{Z}\}$ denote the set of all points adjacent to 0. Let further*

$$U_1(x) := \{\triangle \in \mathcal{T} \mid \textit{all corner points of } \triangle \textit{ are contained in } x + \mathcal{N} + \mathcal{N}\} \setminus U_0(x)$$

denote the "second layer" of triangles around x. In the special case $x = 0$, we abbreviate $U_0 := U_0(0)$ and $U_1 := U_0(1)$.

Lemma 4.1 (Heydenreich, Merkl, Rolles; Lemma 2.6 in [HMR13])**.** *There is a $c_4 > 0$ such that for all $\omega \in \Omega_{N,l}$ with $\omega(0) = \bigcirc$, one has*

$$\sum_{\triangle \in U_0} \mathrm{dist}(\nabla\hat{\omega}(\triangle), \mathrm{SO}(2))^2 \leq c_4 \sum_{\triangle \in U_1} \mathrm{dist}(\nabla\hat{\omega}(\triangle), \mathrm{SO}(2))^2.$$

We omit the proof of the lemma, and prove the analogue to *Theorem 2.1* with defects.

Theorem 4.2. *For ϵ sufficiently small, the constant value of the Jacobian $\nabla\hat{\omega}(\triangle)$ on the set $\triangle \in \mathcal{T}_N$ satisfies*

$$\lim_{l \downarrow 1} \sup_{N \in \mathbb{N}} \sup_{\triangle \in \mathcal{T}_N} E_{P_{N,m,l}}[\ |\nabla\hat{\omega}(\triangle) - \mathrm{Id}|^2\ ] = 0$$

with any norm $|\cdot|$ on $\mathbb{R}^{2\times 2}$.

In general, the model parameter ϵ has to be chosen smaller in *Theorem 4.2* than in *Theorem 2.1*. This choice of ϵ is also the main idea of the proof.

Proof. Let $\omega \in \Omega_{N,l}$ and $x \in I_N$ with $\omega(x) = \hexagon$. From *Lemma 3.1* and *Lemma 4.1* we conclude that there is a constant $c_{41} > 0$ which does not depend on ω and

$$\sum_{\triangle \in U_0(x)} \sum_{x \neq y \in S(\triangle)} (|\hat{\omega}(x) - \hat{\omega}(y)| - 1)^2 \quad \leq \quad c_{41} \sum_{\triangle \in U_1(x)} \sum_{x \neq y \in S(\triangle)} (|\omega(x) - \omega(y)| - 1)^2.$$

Thus we can choose ϵ so small that the side length deviations $(|\hat{\omega}(x) - \hat{\omega}(y)| - 1)^2$ become small enough for inequality (3.7) to be fulfilled for all triangles in $\mathcal{T}_N$, even for the ones which are absent. The rest of the proof is identical to the proof of *Theorem 2.1* if we replace ω by $\hat{\omega}$ in *Lemma 3.2*; this lemma remains true in the case of defects if ϵ is small enough to apply *Lemma 3.1*. To proceed in the same way, we observe that the random variables $\nabla(\hat{\omega}(\triangle))$ are again identically distributed for all $\triangle \in \mathcal{T}_N$ under the new measure $P_{N,m,l}$ as defined in (4.5). This fact is a consequence of the shift- and reflectional invariance of $P_{N,m,l}$. □

Theorem 4.2 includes *Theorem 2.1*, since applying *Theorem 4.2* to $P_{N,m=\infty,l}$ and using the translation ψ from (4.2) and the translational invariance of the Jacobian $\nabla\hat{\omega}(\triangle)$, we obtain the theorem for the push-forwards $P^0_{N,m=\infty,l} = P_{N,l}$. Note that in this case, ϵ can be chosen as in the proof of *Theorem 2.1*. This observation is the object of the following corollary.

Corollary 4.3. *For ϵ sufficiently small and for all $m \in \mathbb{R} \cup \{\infty\}$ one has*

$$\lim_{l \downarrow 1} \sup_{N \in \mathbb{N}} \sup_{\triangle \in \mathcal{T}_N} E_{P^0_{N,m,l}}[\ |\nabla\hat{\omega}(\triangle) - \mathrm{Id}|^2\ ] = 0 \tag{4.6}$$

with the push-forward $P^0_{N,m,l} = \psi[P_{N,m,l}]$, which is given by

$$P^0_{N,m,l}(\mathrm{d}\omega) = \frac{1}{Z^0_{N,m,l}} e^{-H_{N,m}(\omega)} \mathbb{1}_{\Omega^0_{N,l}}(\omega) \mu^0_N(\mathrm{d}\omega) \tag{4.7}$$

where $\mu^0_N = \left\{ \left(\delta_0^{\{0\}} \otimes (\lambda + \delta_{\hexagon})^{\{\tau\}}\right) \Big|_{\{\omega(0) \neq \hexagon\}} + \left(\delta_{\hexagon}^{\{0\}} \otimes \delta_0^{\{\tau\}}\right) \Big|_{\{\omega(0) = \hexagon\}} \right\} \otimes (\lambda + \delta_{\hexagon})^{I_N \setminus \{0,\tau\}}$,
and the partition sum $Z^0_{N,m,l}$ is given by

$$Z^0_{N,m,l} = \int_{\Omega^0_{N,l}} e^{-H_{N,m}(\omega)} \mathbb{1}_{\Omega^0_{N,l}}(\omega) \mu^0_N(\mathrm{d}\omega). \tag{4.8}$$

If $m = \infty$ then $P^0_{N,m=\infty,l} = P_{N,l}$, and we can choose ϵ as in Theorem 2.1.

Proof. We show that $P^0_{N,m,l}$ is in fact as given in (4.7). The other part of the corollary follows from *Theorem 4.2* because the Jacobian is invariant under the translation ψ. Note that ψ is defined $P_{N,m,l}$-almost surely. The Hamiltonian $H_{N,m,l}$ is invariant under ψ, therefore, it suffices to show that the push-forward of the restricted measure $\mu_N|_{\Omega_{N,l}}$ by ψ is given by $\mu^0_N|_{\Omega^0_{N,l}}$ times a positive constant. From $\psi(\Omega_{N,l}) = \Omega^0_{N,l}$ and from Fubini's Theorem, we conclude that

$$\psi[\mu_N|_{\Omega_{N,l}}] = \lambda(\Lambda_{lN})\ \mu^0_N|_{\Omega^0_{N,l}}$$

where $\lambda(\Lambda_{lN}) = (lN)^2 > 0$. Since $P^0_{N,m,l}$ is a probability measure, we obtain (4.7). □

In the next section, we are going to show tightness of $(P^0_{N,m=\infty,l})_{N\in\mathbb{N}}$. A similar proof applies to $(P^0_{N,m,l})_{N\in\mathbb{N}}$. Likewise, we could carry out every result in Section 5 also with the measures $P_{N,m,l}$ and an appropriate definition of Gibbs measures in the case of defects. In this thesis we only discuss infinite-volume arguments in the case without defects.

5 Infinite-volume limit

5.1 Infinite-volume Gibbs measures

In this section we work again with configurations without defects like in Section 2; definitions of $\Omega = \{\omega : I \to \mathbb{R}^2\}$, $\Omega_{N,l}^{per}$, $\Omega_{N,l}$, Ω_∞ and especially that of $P_{N,l}$ are given as in Section 2.

The set of all admissible configurations, Ω_∞, should be thought of as the whole state space. The subspaces $\Omega_{N,l}^{per}$ of configurations with periodic boundary conditions are useful to prove finite-volume theorems like *Proposition 2.5* or *Theorem 2.1.* One general interest of Statistical Mechanics, however, aims at infinite-volume measures on Ω. Now, we show that for all $\delta > 0$, there is a probability measure P on $(\Omega, \mathcal{F})$ that is supported on the set Ω_∞ and satisfies the inequality

$$\sup_{\triangle \in \mathcal{T}} E_P[\ |\nabla \hat{\omega}(\triangle) - \mathrm{Id}|^2\ ] \leq \delta$$

from *Theorem 2.2*. On the other hand, conditioned on the complement Λ^c of any finite subset $0 \in \Lambda \subset I$, the measure P has analogous distribution to $P_{N,l}$. This is achieved by passing from a subsequence $(P_{N_k,l})_{k\in\mathbb{N}}$ to an infinite-volume limit, using tightness arguments. Subsequently, we will verify that this limit has the desired conditional distributions and also satisfies the previous inequality. Proofs in this section are very similar to the proofs in [MR09, Section 4]. We are merely going to show that the same arguments, with minor changes, also suit the measures $P_{N,l}$ from this thesis.

We define Gibbs measures in analogy with the definitions in [MR09, Section 2]. Throughout this section, $\Lambda \subset I$ will denote a finite set. The outer boundary set $\partial\Lambda$ is the set $\partial\Lambda = \overline{\Lambda} \setminus \Lambda$ with the closure

$$\overline{\Lambda} = \{x \in I \mid \exists y \in \Lambda : |x - y| \leq 1\}.$$

We denote the set of triangles with corners in $\overline{\Lambda}$ by $\mathcal{T}(\overline{\Lambda})$, this is to say $\mathcal{T}(\overline{\Lambda}) = \{\triangle \in \mathcal{T} \mid \mathcal{S}(\triangle) \subset \overline{\Lambda}\}$. The set $U_{\overline{\Lambda}}$ stands for the closure of $\cup\mathcal{T}(\overline{\Lambda})$.

For $\Gamma \subset I$ and $\omega \in \Omega$, let ω_Γ be the restriction $\Gamma \to \mathbb{R}^2$, $x \mapsto (\omega(x))_{x\in\Gamma}$. For a subset $A \subset \Omega$, the Γ-restrictions in A are denoted by $A_\Gamma = \{\omega_\Gamma \mid \omega \in A\}$. Furthermore, define for ω_{Γ_1} and $\tilde{\omega}_{\Gamma_2}$ with $\Gamma_1 \cap \Gamma_2 = \emptyset$ the map $\omega_{\Gamma_1}\tilde{\omega}_{\Gamma_2} : \Gamma_1 \cup \Gamma_2 \to \mathbb{R}^2$ to be the extension of ω_{Γ_1} and $\tilde{\omega}_{\Gamma_2}$. Hence we have the equality $\omega_\Lambda\omega_{\Lambda^c} = \omega$. We define the *reference measure*, ν_Λ, on Ω_Λ to be

$$\nu_\Lambda = \begin{cases} \delta_0 \otimes \lambda^{\Lambda\setminus\{0\}} & \text{if } 0 \in \Lambda \\ \lambda^\Lambda & \text{if } 0 \notin \Lambda \end{cases}$$

where λ denotes the Lebesgue measure on $\mathbb{R}^2$. For $\omega_\Lambda \in \Omega_\Lambda$, $\omega_{\partial\Lambda} \in \Omega_{\partial\Lambda}$, and a finite set $\Lambda \subset I$ with $0 \in \Lambda$, define the *local Hamiltonian with boundary condition $\omega_{\partial\Lambda}$* to be

$$H_\Lambda(\omega_\Lambda|\omega_{\partial\Lambda}) = \begin{cases} 0 & \text{if } \omega_\Lambda\omega_{\partial\Lambda} \in \Omega_{\infty,\overline{\Lambda}} \\ \infty & \text{if } \omega_\Lambda\omega_{\partial\Lambda} \notin \Omega_{\infty,\overline{\Lambda}} \end{cases}$$

where $\Omega_{\infty,\overline{\Lambda}}$ denotes the set of *admissible configurations on* $\overline{\Lambda}$ *with boundary conditions on* $\partial\Lambda$. More precisely, $\omega_{\overline{\Lambda}} \in \Omega_{\infty,\overline{\Lambda}}$ for some $\omega \in \Omega$ if and only if

(i) $|\omega(x) - \omega(y)| \in (1, 1+\epsilon)$ for all $x \in \Lambda$ and $y \in \overline{\Lambda}$ with $(x, y) \in E$;

(ii) the map $\hat{\omega}|_{U_{\overline{\Lambda}}} : U_{\overline{\Lambda}} \to \mathbb{R}^2$ is injective, and

(iii) the map $\hat{\omega}$ is orientation preserving on $U_{\overline{\Lambda}}$, this is to say that $\det(\nabla\hat{\omega}(\triangle)) > 0$ for all $\triangle \in \mathcal{T}\left(\overline{\Lambda}\right)$.

The *partition sum*, associated with this local Hamiltonian, is given by

$$Z_\Lambda(\omega_{\partial\Lambda}) = \int_{\Omega_\Lambda} e^{-H_\Lambda(\omega_\Lambda|\omega_{\partial\Lambda})}\nu_\Lambda(\mathrm{d}\omega_\Lambda) \tag{5.1}$$

with the convention $e^{-\infty} = 0$. Because of the parameter ϵ in the definition of $\Omega_{\infty,\overline{\Lambda}}$, the partition sum is finite for all $\omega_{\partial\Lambda} \in \Omega_{\partial\Lambda}$. We remark that $Z_\Lambda(\omega_{\partial\Lambda}) = 0$ is possible for some $\omega_{\partial\Lambda} \in \Omega_{\partial\Lambda}$. In this definition we integrate over the Boltzmann factor e^{-H_Λ}; we could also integrate over the characteristic function $\mathbb{1}_{\Omega_{\infty,\overline{\Lambda}}}(\omega_\Lambda\omega_{\partial\Lambda})$, but we hold to the formalism of Statistical Mechanics at this point. However, we might write it with the characteristic function $\mathbb{1}_{\Omega_{\infty,\overline{\Lambda}}}(\omega_\Lambda\omega_{\partial\Lambda})$ later on.

For $\omega_{\partial\Lambda} \in \Omega_{\partial\Lambda}$ with $Z_\Lambda(\omega_{\partial\Lambda}) > 0$, we define the *finite-volume Gibbs measure,* $P_\Lambda(\,\cdot\,|\omega_{\partial\Lambda})$, *with boundary condition* $\omega_{\partial\Lambda}$ by the formula:

$$P_\Lambda(\mathrm{d}\omega_\Lambda|\omega_{\partial\Lambda}) = \frac{e^{-H_\Lambda(\omega_\Lambda|\omega_{\partial\Lambda})}}{Z_\Lambda(\omega_{\partial\Lambda})}\nu_\Lambda(\mathrm{d}\omega_\Lambda). \tag{5.2}$$

A probability measure P on $(\Omega, \mathcal{F})$ is called an *(infinite-volume) Gibbs measure* if it satisfies the *DLR-conditions*:

$$P(A|\omega_{\Lambda^c}) = \frac{1}{Z_\Lambda(\omega_{\partial\Lambda})}\int_{\Omega_\Lambda} \mathbb{1}_A(\tilde{\omega}_\Lambda\omega_{\Lambda^c})e^{-H_\Lambda(\tilde{\omega}_\Lambda|\omega_{\partial\Lambda})}\nu_\Lambda(\mathrm{d}\tilde{\omega}_\Lambda) \quad P\text{-a.s.} \tag{5.3}$$

for any $A \in \mathcal{F}$ and finite $\Lambda \subset I$ containing $0 \in I$. In particular, this definition includes the requirement $Z_\Lambda(\omega_{\partial\Lambda}) > 0$ for P-almost all $\omega \in \Omega$.

5.2 Proof of Theorem 2.2

Now, we prove that finite-dimensional margins of $(P_{N,l})_{N\in\mathbb{N}}$ are tight, and this sequence has a subsequence whose finite-dimensional margins converge weakly. The following lemma is analogue to [MR09, Lemma 4.1].

Lemma 5.1. *The finite dimensional-marginal distributions of $(P_{N,l})_{N\in\mathbb{N}}$ are tight. As a consequence, there is a strictly increasing sequence $(N_k)_{k\in\mathbb{N}}$ of natural numbers such that the finite-dimensional margins of $P_{N_k,l}$ converge weakly to the margins of a limiting distribution P_l on Ω.*

Proof. In the proof we follow the steps from the proof of [MR09, Lemma 4.1]. We have for any finite edge set $F \subset E$:

$$P_{N,l}(\omega(0) = 0 \text{ and } \forall (x, y) \in F : \ |\omega(x) - \omega(y)| < 1 + \epsilon) = 1, \tag{5.4}$$

therefore, we obtain $|\omega(x)| \le \text{dist}(0, x)(1+\epsilon)$ for all $x \in I$ $P_{N,l}$-almost surely with the graph distance $\text{dist}(0, x)$ from 0 to x in the lattice I. Consequently, for every finite $\Lambda \subset I$, there is a compact set $K \subset (\mathbb{R}^2)^\Lambda$ such that $P_{N,l}(\omega_\Lambda \in K) = 1$ for all $N \in \mathbb{N}$, thus finite-dimensional margins of $(P_{N,l})_{N\in\mathbb{N}}$ are tight. By Prokhorov's Theorem, applied to the Polish space $(\mathbb{R}^2)^\Lambda$, we conclude that there is a subsequence of $(P_{N,l})_{N\in\mathbb{N}}$ such that its Λ-margins converge weakly to a distribution on $(\mathbb{R}^2)^\Lambda$. Taking these subsequences $(P_{N_k(M),l})_{k,M\in\mathbb{N}}$ recursively for each finite set $(I_M)_{M\in\mathbb{N}}$ such that $(N_k(M + 1))_{k\in\mathbb{N}}$ is a subsequence of $(N_k(M))_{k\in\mathbb{N}}$ and considering the diagonal sequence $(P_{N_k(k),l})_{k\in\mathbb{N}}$, we obtain by Kolmogorov's Extension Theorem the measure P_l as the extension of the finite-dimensional weak limits. □

In order to show that the measure P_l from *Lemma 5.1* is an infinite-volume Gibbs measure, we show that the finite-dimensional margins of P_l are absolutely continuous with respect to the corresponding reference measures ν_Λ. For this purpose, we need a lower bound on the mass $\delta_0 \otimes \lambda^{I_N\setminus\{0\}}(\Omega_{N,l})$.

Lemma 5.2. *For all $\epsilon \in (0, 1]$ and $l \in (1, 1 + \epsilon)$, there is an $r = r(\epsilon, l) \in (0, 1/2)$ such that for $N \in \mathbb{N}$, we have*

$$\delta_0 \otimes \lambda^{I_N\setminus\{0\}}(\Omega_{N,l}) \ge (\pi r^2)^{|I_N|-1}. \tag{5.5}$$

Proof. For $r > 0$, we define, like in (3.2) in [HMR13], the set of configurations which are close to the scaled standard configuration $\omega_l(x) = lx$:

$$S_{N,l,r} = \{\omega \in \Omega^{per}_{N,l} \mid |\omega(x) - \omega_l(x)| < r \text{ for all } x \in I_N\}. \tag{5.6}$$

For sufficiently small $r > 0$, depending on ϵ and l, we conclude, like in the proof of [HMR13, Lemma 3.1], that $S_{N,l,r} \subset \Omega_{N,l}$. To prove this inclusion, we have to show the properties $(\Omega 1)$–$(\Omega 3)$ for all $\omega \in S_{N,l,r}$. Let us compute for $(x, y) \in E_N$ and $\omega \in S_{N,l,r}$:

$$\begin{aligned} ||\omega(x) - \omega(y)| - l| &= ||\omega(x) - \omega(y)| - |\omega_l(x) - \omega_l(y)|| \\ &\leq |\omega(x) - \omega_l(x)| + |\omega(y) - \omega_l(y)| < 2r. \end{aligned}$$

If we choose $2r < \max\{l - 1, 1 + \epsilon - l\} < 1$, then ω satisfies $(\Omega 1)$. Condition $(\Omega 2)$ is a consequence of the inequality $\langle v, \nabla\hat{\omega}(x)v\rangle > 0$ for all $v \in \mathbb{R} \setminus \{0\}$, and for all $x \in \mathbb{R}^2$ where $\hat{\omega}$ is differentiable. This inequality holds for small enough r since $\nabla\hat{\omega}$ is close to the identity uniformly on $\mathbb{R}^2$. Hence $\hat{\omega}$ is a bijection onto its image. Here we applied a theorem from analysis which states that a $\mathcal{C}^1$-map f from an open convex domain $U \subset \mathbb{R}^n$ into $\mathbb{R}^n$ with $\langle v, \nabla f(x)v\rangle > 0$ for all $v \in \mathbb{R}^n \setminus \{0\}$ and $x \in U$ is a diffeomorphism onto its image. However, $\nabla\hat{\omega}(x)$ is only piecewise differentiable, but on the straight line L connecting $x, y \in \mathbb{R}^2$ with $x \neq y$, there are only finitely many points $z \in \mathbb{R}^2 \cap L$ where the curve $(\hat{\omega}(ty + (1-t)x))_{t\in(0,1)}$ is not differentiable. Assume that $\langle v, \nabla\hat{\omega}(x)v\rangle > 0$ holds whenever $\hat{\omega}$ is differentiable in x. The curve is piecewise linear, and on each of these pieces, the derivative of the curve forms an acute angle with $y - x$, therefore the curve cannot be closed. Thus, the condition $(\Omega 2)$ is satisfied in the case of a sufficiently small r. Furthermore, condition $(\Omega 3)$ is satisfied by ω_l, therefore also by ω if r is sufficiently small. Hence $S_{N,l,r} \subset \Omega_{N,l}$ for some $r \in (0, 1/2)$, and we conclude

$$\delta_0 \otimes \lambda^{I_N\setminus\{0\}}(\Omega_{N,l}) \geq \delta_0 \otimes \lambda^{I_N\setminus\{0\}}(S_{N,l,r}) = (\pi r^2)^{|I_N|-1} \tag{5.7}$$

where the last equality is obtained by integrating over each $\omega(x)$ with $x \neq 0$ which gives a factor πr^2, and considering that $\omega_l(0) = 0$ and that the measure $\delta_0 \otimes \lambda^{I_N\setminus\{0\}}$ fixes $\omega(0) = 0$. □

The following lemma is proved with the same technique as [MR09, Lemma 4.2]. In this proof an upper bound for the entropy is derived which implies that the Λ-margin of the measure $P_{N,l}$ does not concentrate much mass on events with small mass with respect to the reference measure ν_Λ. The next lemma is basically due to F. Merkl and S. Rolles (see [MR09, Lemma 4.2]), we only make minor changes to its proof.

Lemma 5.3. *For all finite, connected $\Lambda \subset I$ with $0 \in \Lambda$ and $l \in (1, 1 + \epsilon)$, there are constants $c_5(\Lambda, \epsilon, l) > 0$ and $c_6(\Lambda, \epsilon) > 0$ such that for all large enough N and measurable $A \subset \Omega_\Lambda$ with $0 < \nu_\Lambda(A) < c_6$, one has*

$$P_{N,l}(\omega_\Lambda \in A) \leq \frac{c_5}{-\log \frac{\nu_\Lambda(A)}{c_6}}.$$

Note that the constants on the right side do not depend on N. This uniform bound is an important capacity of the lemma, which is used in the next corollary.

Proof. We basically quote the proof of [MR09, Lemma 4.2], only making minor changes to it in order to fit the measures $P_{N,l}$.

Let us fix a finite and connected set $\Lambda \subset I$ with $0 \in \Lambda$. Let T be an (undirected) spanning tree of Λ with edge set in E. Here, we can imagine that we work on a quotient space of E or E_N where edges are undirected. Let N be so large that Λ is contained in the box I_N. Let $b + T$ denote the translation of the tree T by $b \in I$ and define $B_{N,\Lambda}$ to be a maximal subset of I_N with the following properties:

(i) $0 \in B_{N,\Lambda}$;

(ii) the sets $b + \Lambda$, $b \in B_{N,\Lambda}$, are pairwise disjoint subsets of I_N.

Note that there is a constant $c_8(\Lambda) > 0$ such that, for all large enough N, one has

$$|B_{N,\Lambda}| \geq c_8 |I_N|. \tag{5.8}$$

Let us now fix such a set $B_{N,\Lambda}$. Furthermore, let us fix an (undirected) spanning tree T_N in E_N of I_N, which contains the trees $b + T$ for all $b \in B_{N,\Lambda}$. Define

$$\tilde{\Omega}_{N,l} = \{\omega \in \Omega^{per}_{N,l} \mid |\omega(x) - \omega(y)| \leq 1 + \epsilon \text{ for all } (x, y) \in T_N\} \text{ and} \tag{5.9}$$

$$\tilde{\Omega}_{\Lambda} = \{\omega \in \Omega_{\Lambda} | \, |\omega(x) - \omega(y)| \leq 1 + \epsilon \text{ for all } (x, y) \in T\} \tag{5.10}$$

where we have identified the spanning trees T_N and T with their edge sets in E_N and E, respectively. We are going to work with the reference measure $\tilde{\nu}_N$:

$$\tilde{\nu}_N(\mathrm{d}\omega) = \frac{\mathbb{1}_{\tilde{\Omega}_{N,l}}(\omega)}{\nu_{I_N}(\tilde{\Omega}_{N,l})} \nu_{I_N}(\mathrm{d}\omega). \tag{5.11}$$

The normalisation is chosen so that $\tilde{\nu}_N$ is a probability measure. Note that $0 < \nu_{I_N}(\tilde{\Omega}_{N,l}) < \infty$, and therefore $\tilde{\nu}_N$ is well defined. Let ρ denote the margin of $\tilde{\nu}_N$ on the set Ω_Λ. Then ρ is given by

$$\rho(\mathrm{d}\omega_\Lambda) = \frac{\mathbb{1}_{\tilde{\Omega}_\Lambda}(\omega_\Lambda)}{\nu_\Lambda(\tilde{\Omega}_\Lambda)} \nu_\Lambda(\mathrm{d}\omega_\Lambda). \tag{5.12}$$

From equation (5.4) we conclude that $\Omega_{N,l} \subset \tilde{\Omega}_{N,l}$ and thus $P_{N,l} << \tilde{\nu}_N$. With $\tilde{\nu}_N$ we can write for the probability measure $P_{N,l}$:

$$P_{N,l}(\mathrm{d}\omega) = \frac{\nu_{I_N}(\tilde{\Omega}_{N,l})}{\nu_{I_N}(\Omega_{N,l})} \mathbb{1}_{\Omega_{N,l}}(\omega)\tilde{\nu}_N(\mathrm{d}\omega) \tag{5.13}$$

with $\nu_{I_N}(\tilde{\Omega}_{N,l}) = \delta_0 \otimes \lambda^{I_N\setminus\{0\}}(\tilde{\Omega}_{N,l}) = (\pi(1+\epsilon)^2)^{|I_N|-1}$. With the convention $\log(0) = -\infty$ and by *Lemma 5.2*, we have

$$\log \frac{\mathrm{d}P_{N,l}}{\mathrm{d}\tilde{\nu}_N} = \log(\mathbb{1}_{\Omega_{N,l}}) + \log \frac{\nu_{I_N}(\tilde{\Omega}_{N,l})}{\nu_{I_N}(\Omega_{N,l})} \leq \log \frac{\nu_{I_N}(\tilde{\Omega}_{N,l})}{\nu_{I_N}(\Omega_{N,l})} \leq \log \frac{(\pi(1+\epsilon)^2)^{|I_N|-1}}{(\pi r^2)^{|I_N|-1}} \tag{5.14}$$

$\tilde{\nu}_N$-almost surely for some $0 < r(\epsilon, l) < 1/2$. Hence, taking the expectations, we obtain

$$\mathbb{E}_{P_{N,l}}\left[\log \frac{\mathrm{d}P_{N,l}}{\mathrm{d}\tilde{\nu}_N}\right] \leq c_7(|I_N| - 1) \leq c_7|I_N| \tag{5.15}$$

with $c_7 = c_7(\epsilon, l) = \log((1+\epsilon)^2/r^2) > 0$, which is an upper bound on the relative entropy per unit volume. Define the function

$$\begin{array}{rrcl} \psi_N : & \Omega^{per}_{N,l} & \to & \Omega_\Lambda^{B_{N,\Lambda}}, \\ & \omega & \mapsto & ((\omega(b+x) - \omega(b))_{x\in\Lambda})_{b\in B_{N,\Lambda}}. \end{array} \tag{5.16}$$

Let $\Pi_{N,l} = \psi_N[P_{N,l}]$ denote the push-forward of $P_{N,l}$ with ψ_N, and let $\Pi^b_{N,l}$, for $b \in B_{N,\Lambda}$, denote its margins on Ω_Λ. Furthermore, let $\zeta_N = \psi_N[\tilde{\nu}_N]$ be the push-forward of $\tilde{\nu}_N$. Note that $\zeta_N = \rho^{B_{N,\Lambda}}$ is a product measure. The relative entropy cannot increase if we replace the measures by their push-forwards with respect to ψ_N. Thus,

$$E_{P_{N,l}}\left[\log \frac{\mathrm{d}P_{N,l}}{\mathrm{d}\tilde{\nu}_N}\right] \geq E_{\psi_N[P_{N,l}]}\left[\log \frac{\mathrm{d}\psi_N[P_{N,l}]}{\mathrm{d}\psi_N[\tilde{\nu}_N]}\right] = E_{\Pi_{N,l}}\left[\log \frac{\mathrm{d}\Pi_{N,l}}{\mathrm{d}\zeta_N}\right]. \tag{5.17}$$

If in the last expression, the measure $\Pi_{N,l}$ is replaced by the product of its margins $\tilde{\Pi}_{N,l} = \prod_{b\in B_{N,\Lambda}} \Pi^b_{N,l}$, the entropy cannot increase. Since the measure $P_{N,l}$ is invariant under the translations $\omega \mapsto \omega(b+x) - \omega(b)$, all marginal distributions $\Pi^b_{N,l}$ of $\Pi_{N,l}$ are equal. Hence,

$$\begin{aligned} E_{\Pi_{N,l}}\left[\log \frac{\mathrm{d}\Pi_{N,l}}{\mathrm{d}\zeta_N}\right] &\geq E_{\tilde{\Pi}_{N,l}}\left[\log \frac{\mathrm{d}\tilde{\Pi}_{N,l}}{\mathrm{d}\zeta_N}\right] \\ &= \sum_{b\in B_{N,\Lambda}} E_{\Pi^b_{N,l}}\left[\log \frac{\mathrm{d}\Pi^b_{N,l}}{\mathrm{d}\rho}\right] = |B_{N,\Lambda}| E_{\Pi^b_{N,l}}\left[\log \frac{\mathrm{d}\Pi^b_{N,l}}{\mathrm{d}\rho}\right] \end{aligned} \tag{5.18}$$

for any $b \in B_{N,\Lambda}$. The last inequality together with (5.15) and (5.17) yields

$$c_7|I_N| \geq |B_{N,\Lambda}| E_{\Pi^b_{N,l}}\left[\log \frac{\mathrm{d}\Pi^b_{N,l}}{\mathrm{d}\rho}\right] \tag{5.19}$$

for any $b \in B_{N,\Lambda}$. Consequently, from this inequality and from (5.8), we obtain

$$\frac{c_7}{c_8} \geq E_{\Pi^b_{N,l}}\left[\log \frac{\mathrm{d}\Pi^b_{N,l}}{\mathrm{d}\rho}\right] \tag{5.20}$$

for every large enough N.

For a given measurable set $A \subset \Omega_\Lambda$ of probability $0 < \rho(A) < 1$, we conclude with the push-forward measures $\mathbb{1}_A[\Pi^b_{N,m,l}]$ and $\mathbb{1}_A[\rho]$ that

$$\begin{aligned}
&E_{\Pi^b_{N,l}}\left[\log \frac{\mathrm{d}\Pi^b_{N,l}}{\mathrm{d}\rho}\right] \geq E_{\mathbb{1}_A[\Pi^b_{N,l}]}\left[\log \frac{\mathrm{d}\mathbb{1}_A[\Pi^b_{N,l}]}{\mathrm{d}\mathbb{1}_A[\rho]}\right] \\
&= \Pi^b_{N,l}(A) \log \frac{\Pi^b_{N,l}(A)}{\rho(A)} + \Pi^b_{N,l}(A^c) \log \frac{\Pi^b_{N,l}(A^c)}{\rho(A^c)} \\
&\geq -\frac{2}{e} - \Pi^b_{N,l}(A) \log \rho(A) - \Pi^b_{N,l}(A^c) \log \rho(A^c) \geq -\frac{2}{e} - \Pi^b_{N,l}(A) \log \rho(A)
\end{aligned}$$

where the inequality $x \log x \geq -\frac{1}{e}$ (for $x > 0$) was used. Hence, by (5.20) and (5.12), we have

$$P_{N,l}(\omega_\Lambda \in A) = \Pi^b_{N,l}(A) \leq \frac{c_5}{-\log \rho(A)} \leq \frac{c_5}{-\log \frac{\nu_\Lambda(A)}{c_6}} \tag{5.21}$$

with $c_5 = c_7/c_8 + 2/e$ and $c_6 = \nu_\Lambda(\tilde{\Omega}_\Lambda)$. □

As a consequence, we obtain that the finite-dimensional margin of the limiting distribution P_l on $\Lambda \subset I$ is absolutely continuous with respect to the reference measure ν_Λ.

Corollary 5.4. *Let P_l be the infinite-volume limiting distribution from* Lemma 5.1. *Its finite-dimensional margins $P_{\Lambda,l} = P_l(\omega_\Lambda \in \cdot\ )$ are absolutely continuous with respect to the reference measures ν_Λ. Furthermore, $P_l(\Omega_\infty) = 1$, and thus $Z_\Lambda(\omega_{\partial\Lambda}) > 0$ for P_l-almost all $\omega \in \Omega$.*

Proof. A very similar proof is due to F. Merkl and S. Rolles and was published in [MR09, Corollary 4.3]; we borrow the main ideas from it. Take a zero set $Q \subset \Omega_\Lambda$ with respect to the reference measure ν_Λ. For every $\delta > 0$, there is an open set $A_\delta \subset \Omega_\Lambda$ such that $Q \subset A_\delta$ and $0 < \nu_\Lambda(A_\delta) \leq \delta$. Since, by *Lemma 5.1*, the margins of some subsequence $(P_{N_k,l})_{k\in\mathbb{N}}$ converge weakly to the corresponding margins of P_l, we conclude from *Lemma 5.3* in the last inequality that

$$P_l(\omega_\Lambda \in Q) \leq P_l(\omega_\Lambda \in A_\delta) \leq \liminf_{k\to\infty} P_{N_k,l}(\omega_\Lambda \in A_\delta) \leq \frac{c_5}{-\log \frac{\nu_\Lambda(A_\delta)}{c_6}}$$

from which, by letting $\delta \to 0$, we obtain absolute continuity. Note that c_5 and c_6 do not depend on N. We remark that *Lemma 5.3* was only proved for Λ with $0 \in \Lambda$, but choosing I_N large enough such that $\Lambda \subset I_N$, the statement follows for any finite Λ.

In order to show $Z_\Lambda(\omega_{\partial\Lambda}) > 0$ for P_l-almost all $\omega \in \Omega$, we show $P_l(\Omega_\infty) = 1$. The set

$$\{\omega_{I_N} \in \Omega_{I_N} \mid \forall x, y \in I_N : |x - y| = 1 \to |\omega_{I_N}(x) - \omega_{I_N}(y)| \in [1, 1 + \epsilon]\} \subset (\mathbb{R}^2)^{I_N}$$

is closed, and the set

$$\{\omega_{I_N} \in \Omega_{I_N} \mid \exists x, y \in I_N : |x - y| = 1 \text{ and } |\omega_{I_N}(x) - \omega_{I_N}(y)| \in \{1, 1 + \epsilon\}\} \subset (\mathbb{R}^2)^{I_N}$$

has measure zero with respect to ν_{I_N}, and therefore, also with respect to the margin of P_l on Ω_{I_N}. This implies that (Ω1) holds P_l-almost surely. Under the measure ν_{I_N}, the event

$$\{\omega_{I_N} \in \Omega_{I_N} \mid \exists \triangle \in \mathcal{T}_N : \det(\nabla\hat{\omega}(\triangle)) = 0\} \subset (\mathbb{R}^2)^{I_N} \tag{5.22}$$

has measure zero; hence by the continuity of the determinant, property (Ω3) holds P_l-almost surely. We obtain (Ω2) P_l-almost surely as follows. Note that $\mathrm{conv(I_N)}$ is a proper subset of U_N. The boundary of

$$\{\omega \in \Omega_{N,l}^{per} \mid \hat{\omega}|_{\mathrm{conv(I_N)}} \text{ is injective}\} \subset (\mathbb{R}^2)^{I_N} \tag{5.23}$$

is a zero set with respect to ν_{I_N}. This observation is true because injectivity can only be violated if and only if the images of two closed triangles overlap in more than a common edge. Therefore, the boundary of (5.23) is a subset of the set where two triangles overlap in some non-common boundary points. This case can only happen if there are three different vertices in I_N which lie on the same line, but this is a zero set with respect to ν_{I_N}. As a consequence, the set

$$\{\omega \in \Omega \mid \hat{\omega}|_{\mathrm{conv(I_N)}} \text{ is injective}\} \subset (\mathbb{R}^2)^I$$

has measure one with respect to P_l for all $N \in \mathbb{N}$. Thus $\hat{\omega} : \mathbb{R}^2 \to \mathbb{R}^2$ is injective P_l almost surely, and as a result we have $P_l(\Omega_\infty) = 1$.

For all $\omega \in \Omega_\infty$ the partition sum $Z_\Lambda(\omega_{\partial\Lambda})$ is positive, which completes the proof. $\square$

The following theorem states that the infinite-volume limiting distributions P_l from *Lemma 5.1* are infinite-volume Gibbs measures in the sense of equation (5.3).

Theorem 5.5. *For all $l \in (1, 1+\epsilon)$, there is an infinite-volume Gibbs measure, P_l, and a subsequence of $(P_{N,l})_{N\in\mathbb{N}}$ such that all finite-dimensional margins of the measures in the subsequence converge weakly to the corresponding finite-dimensional margins of the Gibbs measure P_l.*

Proof. We only have to show that the measures P_l from *Lemma 5.1* satisfy the DLR-conditions (5.3). We are going to follow the steps in the proof of [MR09, Theorem 2.1]. Let us fix a finite set $\Lambda \subset I$ which contains $0 \in I$. By construction we know the DLR conditions (5.3) for the finite-volume Gibbs measure $P_{N,l}$ instead of P_l. For an arbitrary but fixed $\Sigma \subset I$ with $\Lambda \cup \partial\Lambda \subset \Sigma$, we fix $N \in \mathbb{N}$ large enough for $\Sigma \subset I_N$ to hold. Then for any bounded and continuous function $f : \Omega_\Sigma \to \mathbb{R}$, we can write with the margin $P_{N,l,\Sigma}$ of $P_{N,l}$ on Ω_Σ:

$$\begin{aligned}&\int_{\Omega_\Sigma} Z_\Lambda(\chi_{\partial\Lambda}) f(\chi_\Sigma) P_{N,l,\Sigma}(\mathrm{d}\chi_\Sigma)\\ &= \int_{\Omega_\Sigma}\int_{\Omega_\Lambda} f(\omega_\Lambda \chi_{\Sigma\setminus\Lambda}) e^{-H_\Lambda(\omega_\Lambda|\chi_{\partial\Lambda})} \nu_\Lambda(\mathrm{d}\omega_\Lambda) P_{N,l,\Sigma}(\mathrm{d}\chi_\Sigma)\end{aligned}$$

because for all $\chi \in \Omega_{N,l}^{per}$ and $\omega_\Lambda \in \Omega_\Lambda$, we have

$$\mathbb{1}_{\Omega_{\infty,\overline{\Lambda}}}(\omega_\Lambda \chi_{\partial\Lambda}) \mathbb{1}_{\Omega_{N,l}}(\chi) = \mathbb{1}_{\Omega_{N,l}}(\omega_\Lambda \chi_{I_N\setminus\Lambda}) \mathbb{1}_{\Omega_{\infty,\overline{\Lambda}}}(\chi_{\overline{\Lambda}}).$$

Substituting definition (5.1) of the partition sum Z_Λ and subtracting the left from the right side, we obtain

$$\int_{\Omega_\Sigma}\int_{\Omega_\Lambda} (f(\chi_\Sigma) - f(\omega_\Lambda \chi_{\Sigma\setminus\Lambda})) e^{-H_\Lambda(\omega_\Lambda|\chi_{\partial\Lambda})} \nu_\Lambda(\mathrm{d}\omega_\Lambda) P_{N,l,\Sigma}(\mathrm{d}\chi_\Sigma) = 0. \tag{5.24}$$

By the definition of $\Omega_{\infty,\overline{\Lambda}}$ we have $\nu_{\overline{\Lambda}}(\partial\Omega_{\infty,\overline{\Lambda}}) = 0$, therefore, the integrand in (5.24),

$$\Omega_\Lambda \times \Omega_\Sigma \ni (\omega_\Lambda, \chi_\Sigma) \mapsto [f(\chi_\Sigma) - f(\omega_\Lambda \chi_{\Sigma\setminus\Lambda})] \mathbb{1}_{\Omega_{\infty,\overline{\Lambda}}(\omega_\Lambda \chi_{\partial\Lambda})}, \tag{5.25}$$

is almost everywhere continuous with respect to the reference measure $\nu_\Lambda \otimes \nu_\Sigma$. Because the finite-dimensional margin $P_{l,\Sigma}$ of P_l on the set Ω_Σ is absolutely continuous with respect to ν_Σ by *Lemma 5.4*, we conclude that the function (5.25) is almost everywhere continuous also with respect to the measure $\nu_\Lambda \otimes P_{l,\Sigma}$. The function (5.25) is bounded because f is bounded. Now, letting $N \to \infty$, we conclude from *Lemma 5.1* that

$$\int_{\Omega_\Sigma} \int_{\Omega_\Lambda} (f(\chi_\Sigma) - f(\omega_\Lambda \chi_{\Sigma\setminus\Lambda})) e^{-H_\Lambda(\omega_\Lambda|\chi_{\partial\Lambda})} \nu_\Lambda(\mathrm{d}\omega_\Lambda) P_{l,\Sigma}(\mathrm{d}\chi_\Sigma) = 0. \tag{5.26}$$

By the P_l-almost sure positivity of the partition sum (5.1) from *Corollary 5.4*, the equation (5.26) is equivalent to the DLR conditions (5.3) because Λ and f were arbitrary. □

We have made every preparation for the proof of *Theorem 2.2.* We are going to show that, choosing $l > 1$ sufficiently small, the property in *Theorem 2.1* transfers to the limiting distribution P_l from *Theorem 5.1.*

Proof of Theorem 2.2. Let $\delta > 0$ be arbitrary. By *Theorem 2.1* we choose $l > 1$ so small that the inequality

$$\sup_{\triangle \in \mathcal{T}_N} E_{P_{N,l}}[\ |\nabla\hat{\omega}(\triangle) - \mathrm{Id}|^2\ ] \leq \delta$$

holds for all $N \in \mathbb{N}$. Let us fix a triangle $\triangle \in \mathcal{T}$. Like in the proof of *Theorem 5.1*, we can find a compact set $K \subset (\mathbb{R}^2)^{S(\triangle)}$ such that

$$P_{N,l}(\ (\omega(y))_{y \in S(\triangle)} \in K\ ) = 1$$

for all $N \in \mathbb{N}$. Consequently, the continuous map $\Omega \to \mathbb{R}$, $\omega \mapsto |\nabla\hat{\omega}(\triangle) - \mathrm{Id}|^2$ is $P_{N,l}$-almost surely bounded by the same constant for all $N \in \mathbb{N}$. Therefore, by letting $N \to \infty$ and applying the weak convergence from *Lemma 5.1*, we conclude that

$$E_{P_l}[\ |\nabla\hat{\omega}(\triangle) - \mathrm{Id}|^2\ ] \leq \delta$$

with the infinite-volume Gibbs measure P_l from *Theorem 5.5.* Since the triangle $\triangle \in \mathcal{T}$ is arbitrary, and by $P_l(\Omega_\infty) = 1$ from *Corollary 5.4*, we obtain *Theorem 2.2.* □

In the next corollary, we observe that the inequality $E_P[\ |\nabla\hat{\omega}(\triangle) - \mathrm{Id}|^2\ ] \leq \delta$ enforces a high average point density on the triangle $\triangle$. For a configuration $\omega \in \Omega_\infty$ and a triangle $\triangle \in \mathcal{T}$, define the *point density* $\rho_\triangle(\omega)$ *of* ω *on* $\triangle$ by

$$\rho_\triangle(\omega) = \frac{\pi}{8} \frac{1}{\lambda(\hat{\omega}(\triangle))}.$$

The point density $\rho_\triangle$ describes the filled area per unit volume on the triangle $\hat{\omega}(\triangle)$ in the following sense: imagine that at each corner of $\hat{\omega}(\triangle)$, the centre of a disk with radius

1/2 is located. If the disks do not overlap, then the proportion of the area inside $\hat{\omega}(\triangle)$ that is taken up by the disks, to the area of $\hat{\omega}(\triangle)$ is given by $\rho_\triangle(\omega)$. For the standard configuration $\omega_1(x) = x$, we obtain on every triangle $\triangle \in \mathcal{T}$

$$\rho_\triangle(\omega_1) = \frac{\pi}{2\sqrt{3}} =: \rho_{max},$$

the density of the densest circle packing in the real plane. The density $\rho_\triangle(\omega)$ always satisfies $\rho_\triangle(\omega) \leq \rho_{max}$ if $\omega \in \Omega_\infty$. In the following corollary, we show that the expected density with respect to the Gibbs measure P_l is arbitrary close to the density of the densest circle packing whenever l is chosen sufficiently small.

Corollary 5.6. *For all* $0 < \kappa < 1$, *there is an infinite-volume Gibbs measure* P *such that*

$$\inf_{\triangle \in \mathcal{T}} E_P[\rho_\triangle] \geq \kappa \; \rho_{max}.$$

Proof. For all $\omega \in \Omega_\infty$ and $\triangle \in \mathcal{T}$, we have by (3.2) from *Lemma 3.1* that

$$\max_{x \neq y \in \mathcal{S}(\triangle)} (|\omega(x) - \omega(y)| - 1) \leq \operatorname{dist}(\nabla\hat{\omega}(\triangle), \mathrm{SO}(2)) \leq |\nabla\hat{\omega}(\triangle) - \mathrm{Id}|. \tag{5.27}$$

Fix $\delta > 0$ and define $m = \max_{x \neq y \in \mathcal{S}(\triangle)} |\omega(x) - \omega(y)|$. Since $\lambda(\hat{\omega}(\triangle)) \leq (\sqrt{3}/4)m^2$, we have by (5.27)

$$\rho_\triangle(\omega) \geq \frac{\pi}{2\sqrt{3}} \frac{1}{m^2} \geq \frac{\pi}{2\sqrt{3}} \frac{1}{(|\nabla\hat{\omega}(\triangle) - \mathrm{Id}| + 1)^2}. \tag{5.28}$$

There is a positive constant $c_9(\delta)$ such that for all $x \in \mathbb{R}$

$$(x+1)^2 \leq c_9 x^2 + 1 + \frac{\delta}{2},$$

and thus $(|\nabla\hat{\omega}(\triangle) - \mathrm{Id}| + 1)^2 \leq c_9|\nabla\hat{\omega}(\triangle) - \mathrm{Id}|^2 + 1 + \delta/2$. Choose by *Theorem 2.2* an infinite-volume Gibbs measure P such that $\sup_{\triangle \in \mathcal{T}} E_P[\, |\nabla\hat{\omega}(\triangle) - \mathrm{Id}|^2 \,] \leq \delta/(2c_9)$. Now, the corollary follows from Jensen's inequality applied to the function $x \to 1/(x + 1 + \delta/2)$, which is convex for nonnegative x. By (5.28) one has

$$\begin{aligned} E_P[\rho_\triangle(\omega)] &\geq E_P\left[\frac{\pi}{2\sqrt{3}} \frac{1}{(|\nabla\hat{\omega}(\triangle) - \mathrm{Id}| + 1)^2}\right] \geq \frac{\pi}{2\sqrt{3}} E_P\left[\frac{1}{c_9|\nabla\hat{\omega}(\triangle) - \mathrm{Id}|^2 + 1 + \delta/2}\right] \\ &\geq \frac{\pi}{2\sqrt{3}} \frac{1}{E_P[c_9|\nabla\hat{\omega}(\triangle) - \mathrm{Id}|^2] + 1 + \delta/2} \geq \frac{\pi}{2\sqrt{3}} \frac{1}{\delta + 1}. \end{aligned}$$

□

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Erklärung

Hiermit versichere ich, dass ich die vorliegende Arbeit selbstständig verfasst und keine anderen als die angegebenen Quellen und Hilfsmittel benutzt habe, dass alle Stellen der Arbeit, die wörtlich oder sinngemäß aus anderen Quellen übernommen wurden, als solche kenntlich gemacht sind und dass die Arbeit in gleicher oder ähnlicher Form noch keiner Prüfungsbehörde vorgelegt wurde.

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